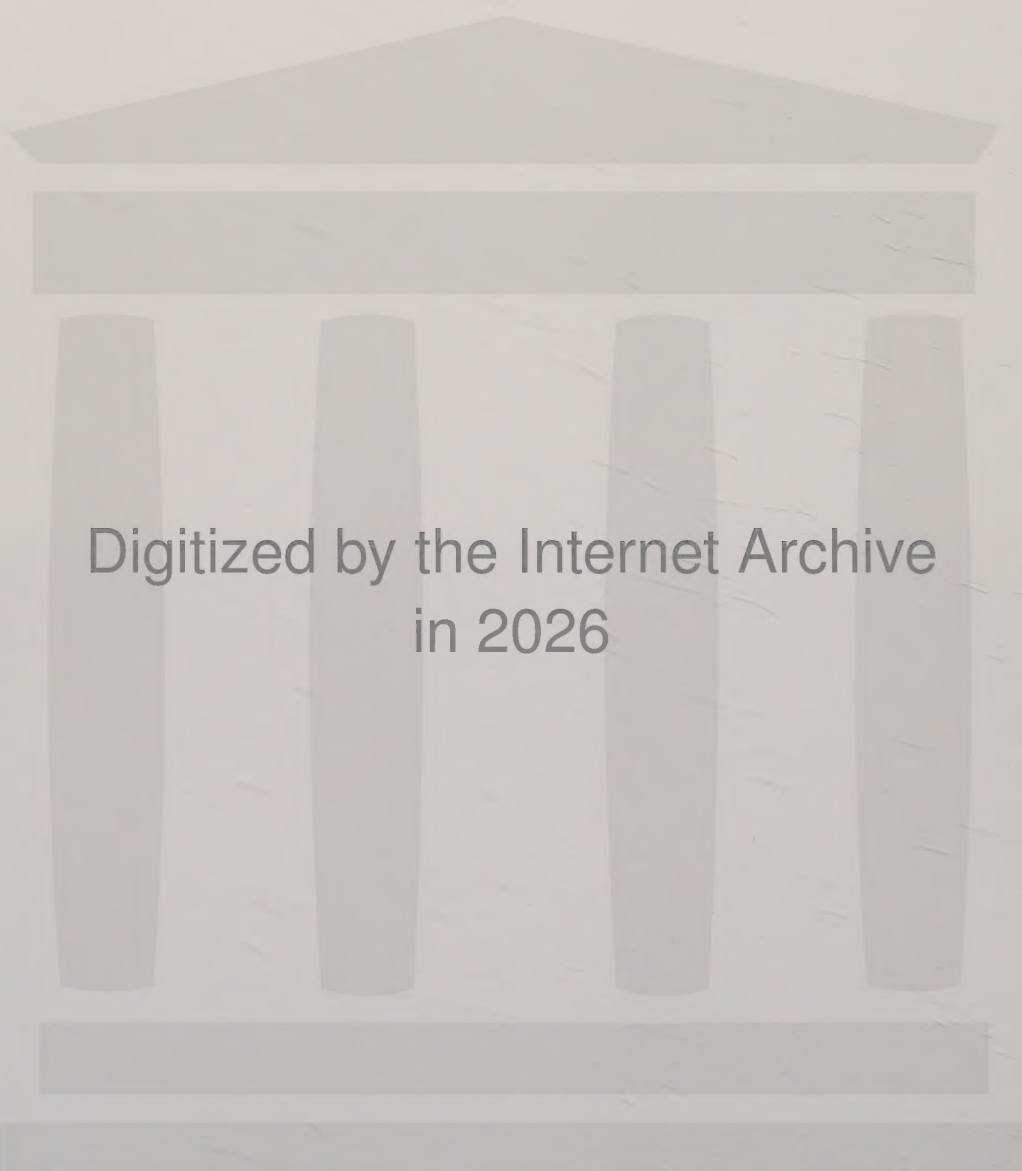


SOCIAL JUSTICE AND FAIR DISTRIBUTIONS

LARS - GUNNAR SVENSSON

**Lund
Economic
Studies**

1977
—



Digitized by the Internet Archive
in 2026

https://archive.org/details/IA41553216_0047

SOCIAL JUSTICE AND FAIR DISTRIBUTIONS

av Lars-Gunnar Svensson



Akademisk avhandling som med tillstånd för avläggande av filosofie doktorsexamen vid Samhällsvetenskapliga fakulteten vid Lunds Universitet framlägges till offentlig granskning

Fredagen den 9 december 1977

kl. 10 i sal 217 i Universitetsbyggnaden

Lund 1977

поправку 2 вносить не надо



SOCIAL JUSTICE AND FAIR DISTRIBUTIONS

by

Lars-Gunnar Svensson



CONTENTS

	page
Preface	i
Chapter I INTRODUCTION	1
1. The Problem of a Social Ranking	1
2. A Discussion of and a Critique on some Wellknown Criteria of Distribution	3
Chapter II ETHICAL PREFERENCES	10
3. A preliminary Definition	10
4. Ethical Preferences Defined over the Utility Space with a Fixed Number of Individuals	18
5. Ethical Preferences Defined over the Utility Space with a Variable Number of Individuals	24
6. Continuous Ethical Preferences	32
7. Ethical Preferences over the Set of Extended Social States	39
Chapter III FAIR DISTRIBUTIONS	41
8. The Form of a Criterion of Distributive Justice	41
9. Some Alternative Social Cakes	47
10. Distributive Justice, a Definition	50
11. Social Rankings	55
12. An Interpretation of a Fair Social State	57
13. Fair Division of a Cake Including one Homogeneous Commodity and Equal Personal Goods	64
14. Fair Division of a Cake Including one Homogeneous Commodity and Personal Goods which may differ between Individuals	74
15. An Interpretation of the Formal Models in sec. 13 and 14 and a Discussion of the Assumptions	78
Chapter IV FAIR DISTRIBUTIONS OF INDIVISIBLES	84
16. Introduction	84
17. A General Model	86

18. The Definition of $\mathcal{E}_2$	88
19. Characterization of Preferences	91
20. A Proof for the Existence of Fair Allocations in $\mathcal{E}_2$	96
21. A Generalization of the Feasibility Set	106
22. Incomplete Preferences	111
23. A Model with Several Variable Characteristics	119
24. A Comparison between Fair Distributions of Indivisibles and a Linear Assignment Problem	128
 Chapter V A FAIR WAGE-STRUCTURE	 132
25. Introduction	132
26. A Fair Wage-Structure in a Formal Model with Completely Indivisible Jobs	134
27. Fair Wages when Preferences are Incomplete	139
28. A Fair Wage-Structure in a Formal Model where Each Job is Associated with a Set of Variable Characteristics	141
29. Fair Wages in a Non Formal Static Model	144
30. Fair Wages in a Non Formal Dynamic Model	148
31. A Comparison between Fair Wages and Some Common Principles of Setting Wages	150
 Bibliography	 155

Preface

There are several objects with this study. As the title indicates, the main purpose is to analyse what shall be meant by a fair welfare distribution in a society. This problem is old in economic theory but in the last thirty years the distributional problems have had to give place to the efficiency problems in an economy and the welfare theory has been developed. Rawls, however, has in his "A Theory of Justice" anew called attention to distributional matters. The definition of a fair welfare distribution or fair social state, is central in this study and can be found in section 10.

The fundamental elements in the definition of fairness are ethical preferences. The definition of ethical preferences, which is used in the definition of fairness, is made in sections 5 and 7. In the rest of chapter II I give an account of the connection between my definition of ethical preferences and existing adjacent concepts e.g. Harsany's ethical preferences or Suppes's "Grading principle of Justice".

The consequences of the definition of fairness are analysed and discussed in different ways. In section 12 an interpretation of the definition is given, and especially the problem of how to identify ethical preferences is being discussed. The existence of fair social states is analysed in two simple formal models; sections 13 and 14.

Within an economic system fairness at different levels is discussed e.g. fairness in public expenditures, fairness in government financing, fairness in wage taxation etc. It is in my opinion very difficult to give a satisfactory definition of fairness in these special cases, without discussing at the same time its relation to fair welfare distributions in general. I think that the former types of fairness merely are means to achieve the later one. As a consequence of this I try to give a definition of fair wages which takes these aspects into consideration, and this is the purpose of chapter V. The

applicable definition of fair wages is given in section 30. Chapter V, A Fair Wage-Structure, is supposed to be an example of how fairness can be defined in the case of "subfairness" as public expenditures, government financing, etc. This is a second aim with this study.

In chapter IV another aim of the study is satisfied. I was originally interested in price equilibrium problems, where assumptions about convexity of preferences and complete divisibility of commodities were not fulfilled. Fair distributions in chapter IV can also be interpreted as a kind of price equilibrium where the commodities are indivisible. The existence problem is examined in this chapter. In section 18 the existence theorem (T1:18) is formulated and the proof requires the space of one section, section 20.

The material can be divided into formal and non-formal parts. The main ideas can be found in the non-formal parts and these correspond primarily to sections 1, 2, 3, 8, 9, 10, 11, 12, 15, 16, 17, 25, 29, 30 and 31. I also want to mention the abbreviations I have used. By A2:18, D1:3, T1:13, and L1:13 e.g. I mean assumption 2 in section 18, definition 1 in section 3, theorem 1 in section 13, and lemma 1 in section 13 respectively.

The problems of distributive justice have not been within the main stream of economic research for a long time. Nevertheless I have received great support of different kinds from my colleagues at the department of economics, University of Lund while working on this study. Of great value has been the continuous encouragement from Professors Björn Thalberg and Bengt Höglund.

Different versions of the study have been discussed at seminars at the department of economics and for comments and constructive critique I want to thank the participants of these meetings. I want particularly to mention the core of the seminars, Professors Björn Thalberg and Bengt Höglund, Anders Borglin, Lars Söderström, Bo Larsson, Göte Hansson, and Rolf Färe. Anders Borglin has also followed the work in detail and closely read the more technical parts of the study and for that I am very grateful. For errors and

shortcomings of this work I am, of course, myself responsible. The idea of using a hypothetical situation for discussions of distributive justice was given to me at a seminar in 1974 about Rawls's "A Theory of Justice", by Professor Ingemar Ståhl. For his support of and interest in my work I am very grateful. I also wish to express my gratitude to Professor Amartya Sen for the discussions and comments as he visited Lund in the spring of 1976. For discussions of non economic aspects of the subject I am thankful to Peter Gärdenfors.

Different drafts have been typed by Kerstin Sundelin, Agneta Kjellgren and the final version by May Nilsson. I am much obliged to them for their typing and interpretation of my handwriting. I am also indebted to Elisabeth Fåk and Robert Cameron who have corrected the language of this last English version.

This study has been supported financially by the Swedish Council for Social Science Research which I also gratefully acknowledge.

Lund, November 1977

Lars-Gunnar Svensson

Chapter I INTRODUCTION

1. The Problem of a Social Ranking

A fundamental property of a society is the mutual advantages which can be achieved by cooperation among the individuals. Therefore, a natural question to ask is: "What is a fair welfare distribution, or what do we mean by a fair social state?" This is a special case, and a case with which we shall deal in the following, of a more general theory in which the problem is to construct a ranking of social states with certain desirable properties.

In economic micro-theory the starting-point is the individual preferences. Although it is not necessary it is natural to let these preferences play a decisive role in a definition of "social preferences". There are many ways, however, in which social preferences can depend on the individual ones and therefore it is important to know which are the possible and which are the desirable ways.

An important theorem illuminating these problems is Arrow's "impossibility theorem".¹

If individual preferences over the set of social states are complete pre-orderings² then there exists no functional relationship f between a social ranking R and the individual rankings $\{R_i\}_i$ if we assume f to have certain properties. More exactly, if we want $R = f(R_1, R_2, \dots, R_N)$, N individuals to be a complete pre-ordering over the set of social states and furthermore want the following four conditions to be fulfilled then such a function f does not exist.

The conditions:

1) (Unrestricted domain.) The domain of f includes all possible individual

1) For proof and discussion see e.g. Arrow (1967) or Sen (1971).

2 Pre-ordering: A reflexive and transitive binary relation.

complete pre-orderings R_i .

- 2) (Pareto-principle.) If x and y are two social states then $xP_i y$ for all i implies xPy .¹
- 3) (Independence of irrelevant alternatives.) If $R = f(R_1, R_2, \dots, R_N)$ and $R' = f(R'_1, R'_2, \dots, R'_N)$ where $\{R_i\}$ and $\{R'_i\}$ are two sets of individual preferences and $R_i = R'_i$ in a subset of the set of social states then $R = R'$ in this subset.
- 4) (Non dictatorship.) There is no individual i such that $xP_i y$ implies xPy for all social states x and y .

Arrow's theorem sets special limits for the use of individual ordinal preferences in constructing social preferences. The conditions are effective in such a way that if we exclude one of them then there exists a social ranking R which fulfils the others.

The modern welfare theory is such an example. In this case there is no claim on social preferences to be complete but xRy for all social states x and y if $xR_i y$ for all individuals i . This is the Pareto-principle. The incompleteness of the principle is of the kind that no distributional considerations are taken into account. It is only possible to compare more or less efficient social states.

We are going to discuss the distribution of welfare in a society and in view of Arrow's theorem we also have to set aside one or several of Arrow's conditions.

Like Arrow we shall start with individual preferences. These preferences will be restricted and preferences fulfilling a symmetry claim will be called ethical preferences. Furthermore we shall only consider incomplete social preferences and these can be of two kinds. One possibility is to define a

¹ $xP_i y$ if $xR_i y$ but not $yR_i x$. P is defined in the same way.

partial social ranking which takes distributional considerations into account and another is just to define the "top" of such a one. In the latter case we shall consider a social state as a division of a "social cake" and shall not try to rank all possible divisions, but only seek for some which with some right can be called fair.

A definition of ethical preferences will be made in chapter two and this is the individual concept of justice. In chapter three a collective concept of distributive justice will be defined, with ethical preferences as a starting point, and thereby a definition of a fair social state will also be given.

2. A Discussion of and a Critique on some Wellknown Criteria of Distribution

Before going any further we shall consider a simple example of a division of a social cake and thereby discuss the divisions recommended by some wellknown criteria.

Consider a "Robinson Crusoe" economy. This economy consists only of two individuals, Crusoe (C) and Friday (F). In this economy there are two positions; to be Crusoe or to be Friday. A position is a complete description of the two individual's situation over the time; how much and with what they shall work, how much commodities of different kinds they dispose, their health, and so on. The positions will be denoted (X,C) and (X,F) where $X = ((X,C), (X,F))$ is the social state corresponding to the two positions.

Assume furthermore that only one commodity needed by these two men has to be produced, e.g. food, and that the only input in this production, which we have to take into account, is the two men's work. In that case there are two commodities, work and food, to be distributed between Crusoe and Friday.

A division of the social cake of the economy is the social state X and thus the social cake is defined by the set of all possible states.

Depending on what the individual positions, and by that also the social cake,

include, we can talk about different kinds of solidarity in the economy. If we e.g. take into account the two men's health when distributing food and work we have a greater solidarity in the economy than if we do not. Thus it is possible to include in the social cake not only goods which can actually be transformed from one individual to another, but also pure individual properties, which we call personal goods, as health, different kinds of abilities and so on.

In our Robinson Crusoe economy we assume that besides work and food even health is included in the social cake and that Crusoe has good health and is able to work but that Friday, with his bad health, is unable to work. This also means that food is the only thing that can be distributed in several ways between Crusoe and Friday. Therefore, we can ask for a fair distribution of food or we can try to rank the possible social states which only differ from each other in the distribution of food.

In economic theory there are different kinds of criteria which solve the distribution-problem and also rank the different possible food distributions.

The utilitarian criterion is formally defined by $U = \sum_{i=1}^N U_i$ where U_i is individual i 's utility and U is the "social utility" in the social state. The existence of an interpersonal cardinal utility scale is assumed and that means that the utility of every position (X,i) in social state X can be measured in one for every individual common and unique utility scale.

Two problems are directly related to the utilitarian criterion. First, there is the question of how to define a unique interpersonal utility scale.

I think that the cardinality is a minor problem, but that interpersonal comparisons are very difficult to make. If you are measuring e.g. length, there are many methods, which all give the same result, to compare two different distances, but there is no simple way to make interpersonal utility comparisons. Of course, this does not necessarily imply the impossibility of such

comparisons.

The second problem, which also depends on the first one, is the special form of the aggregation of individual utilities (by addition) to a social utility.

Why has it to be addition and not some other kind of relation and why shall every individual accept this form? These are questions which are not answered in the Utilitarian theory.

Now assume Crusoe and Friday to agree on an interpersonal cardinal utility scale and on using a utilitarian criterion to determine the distribution of food. Assume further more the following relations to be true:

$$U_C(X, C) = 2 - \frac{1}{x_C}, \quad 0 \leq x_C \leq 1$$

$$U_F(X, F) = 1 - \frac{1}{x_F}, \quad 0 \leq x_F \leq 1.$$

x_C and x_F are Crusoe's and Friday's part of available food and the social state X is determined when the food distribution (x_C, x_F) is fixed. The marginal utility of food is the same $\frac{dU_C}{dx_C} = \frac{1}{x_C^2}$, $\frac{dU_F}{dx_F} = \frac{1}{x_F^2}$ but for equal distribution of food the position (X, C) gets a higher utility than (X, F) according to Friday's bad health.

We see in this case that a social optimum, $\max U = U_C + U_F$, occurs when $x_C = x_F$ i.e. when food is distributed equal in spite of that Crusoe is working but Friday is not. This result depends of course totally on the special form of the utility function, and that also shows that the definition of utility scale is completely decisive for the resulting distribution.

In the choice of utility functions U_C and U_F we have assumed decreasing marginal utility of food. With increasing marginal utility we would have got quite another distribution of food, a distribution where only one of x_C and x_F would be greater than zero.

Thus, there is no point in discussing the utilitarian criterion until the utility scale can be determined in a natural way.

Now suppose Crusoe and Friday of having agreed on using Rawls¹ maximin principle. A formal expression is: $U = \min_{0 < i \leq N} U_i$ where U_i is an ordinal interpersonal measure of the utility of the position of individual i in a social state and U is the "social" utility which has to be maximized.

Even in this case we have the problem of interpersonal comparisons of utility. The ordinal character of the measure does not simplify the problem very much.

We have also the problem of the special form of the social utility U . Even if Rawls argue for this form very penetratingly there are intuitively grave objections against it. The maximin principle implies a very strong claim of equality, differences in the utility level of different positions (X,i) are allowed only if it is not possible to increase the utility of the worst-off individual.

Now consider the Robinson Crusoe economy and assume that it is very difficult to compensate Friday's bad health by giving him much food. If the utility of the positions (X,C) and (Y,F) shall be equal the food has to be distributed in such a way that Crusoe gets extremely little food and perhaps is starving, while Friday gets the most part. The problem in this case is that Friday gets more food than he needs when at the same time Crusoe gets much less than he needs. The positions are in this way equally bad.

It is not quite certain that this extreme food distribution is implied by Rawls' maximin principle. If Crusoe's part of the food available is so small that it restricts his possibilities of working, and thereby possibilities of producing food, then the maximin principle allows the positions to have different utility levels. The reason is that by giving Crusoe more food he will produce so much more that even Friday's utility level can be increased a little.

¹ See Rawls (1972).

The main problem with the principle still remains, however. It does not really take into account the two men's need of food.

The two social utility functions, the utilitarian summation of individual utilities and Rawls' maximin principle, give a complete ranking of all possible social states and by maximizing the functions respectively we get a socially "best" state. This state can be defined as a fair division of the social cake.

Fair division is often related to some kind of equal division, and as we have seen this is also an important property of Rawls' maximin principle. In the utilitarian criterion individuals are instead treated equal in the sense that marginal changes from an optimum in the set of feasible social states affect the utility of different individuals equally much.

Now consider the fairness criterion¹ which is a criterion of equal division i.e. it is not possible to rank all possible divisions of a cake, only to find out the fair ones.

If we assume each individual i to have preferences R_i , e.g. a complete pre-ordering over all possible positions (X,j) , then a fair state X according to the fairness criterion satisfies two conditions:

- 1) The state X is Pareto-optimal.
- 2) $(X,i) R_i (X,j)$ for all individuals i and j , i.e. in the state X there is no envy among the individuals.

Fairness is a simple extension of "equal division" of a homogeneous cake. If, for example, a sum of 100 kr is to be equally divided between two individuals A and B every division (X_A, X_B) with X_A to A and X_B to B and $X_A = X_B$,

$X_A + X_B \leq 100$ is an equal division and is characterized by the fact that no one envies the other. If you, furthermore, demand an efficient division (a

¹ See e.g. Varian (1974).

Pareto-optimal division) the only possibility is $X_A = X_B = 50$. These two properties, freedom from envy and efficiency, are just the ones which define a fair state.

The important difference between the two previous criteria and the fairness criterion is the way in which interpersonal comparisons are made. In the fairness criterion each individual makes his own comparisons and there is no need for a common cardinal or ordinal utility scale.

The fairness criterion, however, defines an equal division of the social cake and therefore the same kind of criticism can be raised against it as has been done against Rawls' maximin principle. Too strong a claim of equality will sometimes imply that too small an account of the individual's needs has been taken.

We can illustrate these objections against total equality at any prize by considering the Robinson Crusoe economy in an initial and a final situation. The final situation is exactly the one we have already considered, and the initial is the situation before Friday had got his bad health. The two men are now equal in every aspect and they know that some accident could make any of them unwell. For that reason they want to make an agreement about food distribution in case an accident should happen. In that case it is not very probable that the agreement would result in such an extreme distribution of food which is recommended by Rawls' maximin principle or by the fairness criterion.

This "insurance" distribution in the final situation, achieved by an agreement in the initial situation, seems to be a more suitable starting point to define social justice than to use the discussed criteria uncritically.

The idea of using a hypothetical or initial situation in which agreements about utility distributions in potential situations are made is old. This is the technique used in the social contract theory as can be found in Locke,

Rousseau, Kant and later in Rawls.¹

Our purpose is to formalise these social contract ideas and thereby use restricted individual preferences, ethical preferences, with which interpersonal comparisons of utility levels are possible to make, to give a definition and an analysis of a fair social state. The problem to solve is how to define an equal division of a social cake which will avoid the problems concerning equal division discussed in the example of the Robinson Crusoe economy.

The examples have shown that it is not always "socially satisfactory" to divide equal. Justice, however, is by definition some kind of equal division, so our problem is: "What is to be divided equal?"

In the next chapter we will define individual distributive justice, ethical preferences, and in chapter II analyse the collective concept of distributive justice.

¹ See Rawls (1972) page 11.

Chapter II ETHICAL PREFERENCES

We will in this chapter define and analyse the concept of ethical preferences. The first two sections (3 and 4) include an extension of already existing concepts and in section 5, 6 and 7 we will define ethical preferences in the form which will be used.

Ethical preferences will be defined over a set of social states or alternatively over a set of corresponding utility distributions.

In sec. 4 we will consider utility distributions for a fixed number of individuals and define ethical preferences in that case. In sec. 5 we also want to compare utility distributions including different numbers of individuals and this will lead us to consider relative distributions.

The consequences of defining a topology on the set of utility distributions will be analysed in sec. 6, and finally we generalize the concept of social state in sec. 7.

3. A Preliminary Definition

Arrow's impossibility theorem has shown the non-existence of a complete social ranking satisfying certain conditions. There are no difficulties, however, in constructing partial rankings, the Pareto principle is such an example. The degree of incompleteness is in a sense depending on the assumptions made on individual preferences. Because of the weak assumptions which are normally made on individual preferences the Pareto principle cannot be used for distributional statements, it is only an efficiency principle.

In Harsanyi (1955) there is a discussion of subjective and ethical preferences.

Ethical preferences are those, which the individual have "when he forces a special impartial and impersonal attitude upon himself". The claim of imper-

sonality of the ethical preferences is the important element of justice and is also a restriction on the preferences. Consequently, the Pareto principle applied to ethical preferences will give a "more" complete ranking of social states than the usual Pareto principle and at the same time take distributional considerations into account.

The subjective preferences on the other hand are "the old ones" i.e. on which no restrictions are put and which are used to rank individual situations.

We shall now give a formal definition of ethical preferences and some primitive concepts are needed. We therefore define the following formal model.

Consider an economy with a finite number n of individuals and let

$I = \{i_1, i_2, \dots, i_n\}$ be the set of individuals. We shall also use the concept position. A position is assumed to be a complete description of everything that is relevant for a valuation of an individual's whole situation. What is relevant depends on what is included in the social cake and this will be discussed in chapter II. Let $\mathcal{P}$ be the set of positions.

We also talk about individual positions and by that mean a correspondence $i \rightarrow X(i) \in \mathcal{P}$, $i \in I$. If $a = X(i) \in \mathcal{P}$ we also denote the individual position: (a, i) or $(X(i), i)$. The difference between a position and an individual position is merely that in the case of individual position we have specified which individual holds a special element in $\mathcal{P}$. The reason for making this difference is that different individuals may have different preferences over the positions, so if two different individuals hold the same position in $\mathcal{P}$ they may value them differently. This is necessary to take into account when interpersonal comparisons are to be made.

By a social state X we mean a function from the set of individuals to the set of positions i.e. $X: I \rightarrow \mathcal{P}$. Thus a social state is a collection of individual positions, one for each individual. Let $\mathcal{S} = \{X: X \text{ is a function } I \rightarrow \mathcal{P}\}$ denote the set of social states.

A couple of examples will illuminate the concepts of position and individual position. First assume the social cake to be divided to consist of a certain amount of commodities.

Then a position is just a commodity bundle, and an individual position a commodity bundle held by a special individual.

Now assume that the social cake will have to include not only commodities, but also individual properties, which we call personal goods such as health, different kinds of abilities and so on. Then a position is: "to hold certain commodities and personal goods" and in the case of individual position we have also specified the individual holding the same.

We now see that the concept position is a simple extension of the concept commodity bundle used in micro economic theory and that a social state corresponds to an allocation.

We shall also assume each individual to have preferences over the set $\mathcal{P}$.

Assumption 1: Each individual $i \in I$ has a complete pre-ordering $\succsim_i'$ (reflexive, transitive binary relation) over the set $\mathcal{P}$ of positions.

It is natural to assume the same preferences to be true for individual positions.

Assumption 2: Each individual $i \in I$ has preferences $\succsim_i''$ over his set of individual positions $\mathcal{P} \times \{i\}$ which coincide with his preferences over $\mathcal{P}$ i.e. for $x, y \in \mathcal{P}$ and $(x, i), (y, i) \in \mathcal{P} \times \{i\}$ we have: $x \succsim_i' y \Leftrightarrow (x, i) \succsim_i'' (y, i)$.

We call these preferences $\succsim_i''$ the individual subjective preferences. These preferences can be extended to the whole set $\mathcal{P} \times I$ and we also assume each individual to have such an extension.

Assumption 3: Each individual $i \in I$ has a complete pre-ordering $\succsim_i$ over the set $\mathcal{P} \times I$ such that $\succsim_i''$ is a subrelation to $\succsim_i$ i.e.

$$(x,i) \succsim_i (y,i) \Leftrightarrow (x,i) \succsim_i (y,i) \text{ for } (x,i), (y,i) \in \mathcal{P}_x \{i\} \subset \mathcal{P}_x I.$$

A3 is an assumption of interpersonal comparisons. We do not assume the existence of one and only one possibility to compare different individuals' utility levels, only that every individual can in some way make these comparisons. It is, for example, possible to define $\succsim_i$ according to $(x,j) \succsim_i (y,k) \Leftrightarrow (x,i) \succsim_i (y,i)$ for $i,j,k \in I$ and $x,y \in \mathcal{P}$ because every individual has preferences over the set of positions $\mathcal{P}$. In that case, however, the individual i does not take into account the preferences of individuals k and j which can be desirable when valuing their own positions.

Another possibility to define $\succsim_i$ is first to rank his own positions (x,i) $x \in \mathcal{P}$ according to his preferences $\succsim_i$ and then e.g. individual one's positions $(x,1)$ according to $\succsim_1$, and then individual two's positions $(x,2)$ according to $\succsim_2$ and so on. In that case we have $(x,i) \succsim_i (y,j) \forall x,y \in \mathcal{P}$ and $\forall j \in I$ and $(x,j) \succsim_i (y,k), j \neq k, j \neq i, k \neq i$ if and only if $j > k$ for any $x,y \in \mathcal{P}$. Now the individual i puts himself at the top and other individual positions are worse than his. However, he takes into account the other individuals' preferences over their own positions.

We see that A1 is a very weak assumption and makes very few restrictions on the interpersonal comparisons. To get a fruitful theory of justice we have to add certain conditions to A1. In general we shall make these supplementary assumptions when special models are analysed and when there is often some kind of "natural" interpersonal comparison norm. For the time being we only implicitly make assumptions about further restrictions on the preferences $\succsim_i$ so that it is meaningful to talk about interpersonal comparisons and e.g. exclude such cases as those where the individual ranks his own position as worse than everybody else's as a rule. If we do not make such assumptions there need not be any significant difference between the preferences $\succsim_i$ and $\succsim_i$.

Possible assumptions on preferences $\succsim_i$ are the axiom of identity and the axiom of complete identity which we define.

Definition 1: Preferences $\succsim_i$ satisfy the axiom of identity if $(x,i), (y,i) \in \mathcal{P} \times I$ and $(x,i) \succsim_i (y,i)$ implies that $(x,i) \succsim_j (y,i)$ for all $j \in I$.

Definition 2: Preferences $\succsim_i$ satisfy the axiom of complete identity if $(x,j), (y,k) \in \mathcal{P} \times I$ and $(x,j) \succsim_i (y,k)$ implies that $(x,j) \succsim_r (y,k)$ for all $r \in I$ i.e. the preferences $\succsim_i$ are the same for all individuals.

Of course there is a great difference between these two axioms. The first one only impose very weak conditions on the preferences $\succsim_i$ by merely claiming the individual's own taste to be taken into account, as his individual positions are ranked. The second one presupposes, on the other hand, a complete unanimity between all individuals on the interpersonal comparisons.

Up to now we have talked about preferences over individual positions $\mathcal{P} \times I$. These preferences, however, may just as well be preferences over the set $\mathcal{S}$ of social states. We define $X \succsim'_i Y$ for $X, Y \in \mathcal{S}$ if and only if $X(i) \succsim_i Y(i)$. The individual ranks social states as he ranks his own position in the states respectively. This does not prevent the individual from being concerned about other individuals' situation in the state, though there are no explicit claims in that direction. We want now to impose such "claims of justice" on the preferences, and therefore we shall define ethical preferences. As a starting-point we take Suppes' "Grading Principle of Justice".¹

Definition 3: According to individual $i \in I$ a social state X "is more just than" a state Y , $X J_i Y$, if there exists a permutation π of the individual set I such that:

$$\begin{aligned} (X(j), j) &\succsim_i (Y(\pi(j)), \pi(j)) \text{ for all } j \in I \text{ and} \\ (X(j), j) &\succ_i (Y(\pi(j)), \pi(j)) \text{ for some } j \in I. \end{aligned}$$

¹ See Sen (1971) page 153.

Theorem 1: Each J_i is an asymmetric and transitive but not necessarily complete binary relation on $\mathcal{Y}$.

This theorem is proved in Sen (1971)¹ and we do not repeat the proof.

The element of justice in the "grading principle" is of course the claim of symmetry in treating different individuals. This can be interpreted in such a way that the ranking of social states is made without any concern about which particular individual will get a better or a worse situation. The "grading principle" corresponds very well to Harsanyi's² claim of impartiality of ethical preferences and therefore we shall take the grading principle as a part of the definition of ethical preferences. Before defining ethical preferences we define what is meant by a symmetrical and strictly monotone pre-ordering.

Definition 4: R_i is a symmetrical and strictly monotone pre-ordering over $\mathcal{Y}$ if for $X, Y \in \mathcal{Y}$ there exist a permutation π of I such that

$(X(j), j) \succsim_i (Y(\pi(j)), \pi(j))$ for all $j \in I$ implies that $X R_i Y$ and

$(X(j), j) \succ_i (Y(\pi(j)), \pi(j))$ for some $j \in I$ implies that $X P_i Y$ where P_i is the strict relation ($X P_i Y \Leftrightarrow X R_i Y$ and not $Y R_i X$).

The pre-ordering R_i is called symmetrical and strictly monotone because if $X \in \mathcal{Y}$ and for a permutation π of I there exists a state $Y \in \mathcal{Y}$ with

$(X(j), j) \sim_i (Y(\pi(j)), \pi(j))$ then $X I_i Y$ ³ i.e., if the utility levels of different individuals are changed in X and Y , but the utility distribution is the same, then X and Y are equally valued. If we, on the other hand, only consider identity permutations $\pi(i) = i$ then the monotonicity property is obvious.

Definition 5: By individual ethical preferences we mean a complete, symmetri-

1 See Sen (1971) page 153.

2 Harsanyi (1955).

3 $X I_i Y \Leftrightarrow X R_i Y$ and $Y R_i X$.

cal and strictly monotone pre-ordering R_i , $i \in I$ over the set of social states $\mathcal{Y}$.

The definition of ethical preferences can be compared to Suppes's "Grading Principle of Justice" J_i and we immediately see that J_i is a subrelation to the strict ethical preferences P_i . In this way ethical preferences are an extension of the Grading Principle.

We have now defined subjective and ethical preferences in A2 and D5. In D5 we impose certain conditions on the preferences, which are not made in A2. Nevertheless, there need not be any significant difference between subjective and ethical preferences if no further conditions on the extension from $\succeq_i$ to $\succeq_i$ are imposed. If we interpret $\succeq_i'$ or $\succeq_i$ as preferences (R_i) over $\mathcal{Y}$, by for $X, Y \in \mathcal{Y}$ putting $X R_i Y$ if $X(i) \succeq_i' Y(i)$ and furthermore $X P_i Y$ if $X(i) \sim_i' Y(i)$ and $X(j) \succeq_i' Y(j) \forall j \in I$ and $X(j) \succ_i' Y(j)$ for some $j \in I$, then we can prove:

Theorem 2: To each set $\{\succeq_i'\}_i$ of subjective preferences over $\mathcal{P}$ there exists a set of extensions $\{\succeq_i\}_i$ to $\mathcal{P} \times I$, such that the subjective preferences R_i over $\mathcal{Y}$ corresponding to $\succeq_i'$ are ethical preferences.

Proof: For $(x, j), (y, k) \in \mathcal{P}$ define $\succeq_i$ according to $(x, j) \succ_i (y, k)$ if $j > k$ and $(x, j) \succeq_i (y, k)$ for $j = k$ if $(x, j) \succeq_j' (y, k)$. We now see that $(X(j), j) \succeq_i (Y(\pi(j)), \pi(j))$, $\forall j \in I$ and for a permutation π of I , is possible only if $\pi(j) = j \forall j \in I$. Thus we have that $(X(j), j) \succeq_i (Y(\pi(j)), \pi(j))$, $\forall j \in I$ and for a permutation π , implies that $\pi(j) = j$ i.e.

$(X(j), j) \succeq_i (Y(j), j) \forall j \in I$. In that case we also have $(X(i), i) \succeq_i' (Y(i), i)$ and $X R_i Y$. If furthermore $(X(j), j) \succ_i (Y(j), j)$ for some $j \in I$ we have $X P_i Y$. This proves that the relation R_i defines ethical preferences. ■

This shows the importance of making further restrictions on the extended subjective preferences $\succeq_i$ in order to get a good theory of justice. If, for instance, the society demands certain members e.g. politicians to use ethical preferences to make impartial decisions this could be worth nothing in view

of T2. By making the interpersonal comparisons in a suitable way, it is always possible to satisfy purely egoistic interests, even if ethical preferences are used.

This weakness of ethical preferences is of course also a weakness of Suppes' Grading Principle of Justice because ethical preferences are an extension of Suppes' principle. There is, however, often some natural interpersonal comparison norm, not quite unique perhaps, but determined within certain limits and when applied it makes a real difference between subjective and ethical preferences.

The intuitive idea of ethical preferences and thereby also an implicit restriction on the extension of subjective preferences can be given by an interpretation. We may regard ethical preferences as those which an individual would have if he knew that he was going to hold one of n positions in a social state. He does not know which one, only that there is equal probability of getting anyone. From this interpretation the symmetrical and strictly monotone property of ethical preferences is natural and rational.

Ethical preferences can also be interpreted as those which the individual would have in Rawls' original position.¹ In this case, however, the individual knows neither his future position nor any probabilities about getting a special position, but the strictly monotone and symmetrical property is still natural.

Ethical preferences and Suppes' Grading Principle of Justice are individual complete and partial respectively, rankings of social states i.e. social preferences. A normal claim on social preferences is compatibility with the Pareto principle. This is not always the case with Suppes' Grading Principle and thereby not with ethical preferences. The axiom of identity, however, solves these problems.

¹ See Rawls (1972) page 17.

Definition 6: A social state X is better than a social state Y according to the weak Pareto strict relation P (XPY) if and only if $X(i) \succ'_i Y(i) \forall i \in I$. $\succ'_i, i \in I$ are the subjective preferences over $\mathcal{P}$ of the individuals.

Theorem 3: When the number of individuals $n \geq 2$, the weak Pareto strict relation P is incompatible with each J_i (Grading Principle), for $i = 1, 2, \dots, n$, for some logically possible set of subjective preferences $\{\succ'_i\}_{i=1}^n$.

This is a theorem in Sen (1971) page 155 and is also proved there. Of course the same theorem is true for ethical preferences instead of J_i because J_i is a subrelation to strict ethical preferences.

If we assume the axiom of identity (or the axiom of complete identity) then $T3$ is not true.

Theorem 4: Under the axiom of identity:

$$\left. \begin{array}{l} (X(i), i) \succsim_i (Y(i), i) \forall i \in I \\ (X(i), i) \succ_i (Y(i), i) \text{ for some } i \in I \end{array} \right\} \Rightarrow$$

$\Rightarrow X P_i Y$ where $X, Y \in \mathcal{S}$ and $P_i, i \in I$ are the strict ethical preferences of the individuals.

Proof: Consider the ethical preferences R_j of individual $j \in I$. In that case we have: $X(i) \succsim'_i Y(i) \forall i \in I \Rightarrow (X(i), i) \succsim_j (Y(i), i) \forall i \in I$ and $(X(i), i) \succ_j (Y(i), i)$ for some $i \in I$ according to the axiom of identity. The symmetrical and strictly monotone property (with $\pi(i) = i \forall i \in I$) of ethical preferences then gives that $X P_j Y$. ■

4. Ethical Preferences Defined over the Utility Space with a Fixed Number of Individuals

We are now going to look a little closer at the concept ethical preferences and thereby separate the symmetrical and strictly monotone property. By using a utility scale instead of the set $\mathcal{S}$ as a starting point, we can get an alternative definition of ethical preferences, which in most cases is equivalent

ent to the former one, but is more fit for extending purposes. We shall later on make an extension of ethical preferences to cases where social states with different numbers of individuals are compared.

Consider once more the set of positions $\mathcal{P}$ and assume $\mathcal{P}$ to be a connected topological space.¹ We now assume that the preferences $\succsim'_i$ over $\mathcal{P}$ can be represented by a continuous utility function. This representation is always possible if the preferences $\succsim'_i$ over $\mathcal{P}$ are continuous and the set $\mathcal{P}$ is separable.²

Assumption 1: The subjective preferences $\succsim'_i$ can be represented by an ordinal continuous utility function $f'_i: \mathcal{P} \rightarrow \mathbb{R}$.

From the definition of an ordinal utility function we have $f'_i(x) \geq f'_i(y) \Leftrightarrow x \succsim'_i y$ for $x, y \in \mathcal{P}$. Because $\mathcal{P}$ is connected and the continuity of f'_i we also know that the range of f'_i is an interval of the real line. If $A_i = \{t : t = f'_i(x) \text{ for some } x \in \mathcal{P}\}$ we define A_i to be the utility scale corresponding to the utility function f'_i . The utility scale A_i is not unique but many possible A_i exist for given preferences $\succsim'_i$.

Now consider the set of individual positions $\mathcal{P} \times I$ and assume $\mathcal{P} \times I$ to be a topological space with the product topology and with the discrete topology on I . Just as in the case of preferences $\succsim'_i$ we assume that preferences $\succsim_i$ over $\mathcal{P} \times I$ can be represented by an ordinal utility function.

Assumption 2: The subjective preferences $\succsim_i$ can be represented by a continuous ordinal utility function $f_i: \mathcal{P} \times I \rightarrow \mathbb{R}$ where $f_i(x, i) = f'_i(x) \forall x \in \mathcal{P}$ and $\forall i \in I$.

We have earlier remarked that further restrictions have to be made on preferences $\succsim_i$ for the interpersonal comparisons to be real. We can list some

1 Simmons (1963) page 143.

2 See Debreu (1964).

possible claims which may be imposed on the function f_i .

i) $A_i = \{f_i(x, j) : x \in \mathcal{S}\} \forall j \in I$.

ii) For $x, y \in \mathcal{P} : f_i(x, i) \geq f_i(y, i) \Leftrightarrow f_j(x, i) \geq f_j(y, i) \forall j \in I$.

iii) For $x, y \in \mathcal{P}$ and $j, k \in I : f_i(x, j) \geq f_i(y, k) \Leftrightarrow f_q(x, j) \geq f_q(y, k)$

$\forall i, q \in I$.

i) is an assumption that the same utility scale can be used for all individuals and that if one particular individual can reach a special utility level then everybody else can do so too. It is, for example, impossible that one individual, under any circumstances, is better or worse off than somebody else.

ii) is the axiom of identity and iii) the axiom of complete identity. i) and ii) or iii) are independent of each other.

In the future we shall assume preferences to have property i) and we state this as a formal assumption.

Assumption 3: The ordinal utility function f_i has property i) for all individuals $i \in I$.

We can now define ethical preferences using the utility scale A_i as a basis. Consider one particular individual, and in order to simplify matters, we drop the individual notation and shall write f instead of f_i , A instead of A_i and so on.

Now let $F : \mathcal{S} \rightarrow U_n$ be a function from social states to the space $U_n = \{u : u \text{ is a function: } I \rightarrow A\}$ of utility distribution functions corresponding to f and defined according to $F(X) = u$ where $u(i) = f(X(i), i)$, $i \in I$. $F(X)$ is the utility distribution in the social state X and n is the number of individuals.

We can now repeat the definitions from the former section and formulate the corresponding concept in the utility space U_n . Thereby we can also separate

the symmetrical and strictly monotone property into two concepts.

Definition 1: R is a symmetrical pre-ordering over U_n if $u \in U_n$ and $u I \pi u$ for all πu and where I is the corresponding indifference relation, $\pi u(i) = u(\pi i)$, $i \in I$ and π is a permutation of the individual set I .

Definition 2: R is a strictly monotone pre-ordering over U_n if $u, v \in U_n$, $u \geq v$, $u \neq v$ implies that $u P v$.

Definition 3: By individual ethical preferences we mean a complete, symmetrical and strictly monotone pre-ordering R over the utility space U_n .

Now we have two definitions of ethical preferences and this is meaningful only if the two definitions are the same in a certain sense. There are, however, some differences. In D3:4 we use a more specified model. Firstly, we have assumed a certain structure on the set of positions $\mathcal{P}$ and besides assumed that the subjective preferences may be represented by a continuous ordinal utility function. Secondly, we put some restriction (A3:4) on the interpersonal comparison which will make a real difference between subjective and ethical preferences i.e. T2:3 is no longer correct. We shall prove this, and that under A1:4, A2:4 and A3:4 ethical preferences according to D5:3 and according to D3:4, in principle, are the same. Because the two definitions of ethical preferences are made over different sets they are not exactly the same, but there is a natural and unique correspondence between them. To each social state $X \in \mathcal{J}$ there is exactly one utility distribution in U_n and a relation over U_n is induced. To one element in U_n , on the other hand, there may be several corresponding social states. We therefore prove:

Theorem 1: Under A1, A2 and A3 the following is true:

- i) Each pre-ordering R over U_n defines a pre-ordering R' over $\mathcal{J}$ according to $X, Y \in \mathcal{J}$ and $X R' Y \Leftrightarrow u R v$, $u = F(X)$, $v = F(Y)$.
- ii) Each pre-ordering R' over $\mathcal{J}$ defines a pre-ordering R over U_n according to $u, v \in U_n$ and $u R v \Leftrightarrow X R' Y$ for some X, Y with $u = F(X)$, $v = F(Y)$.

- iii) Furthermore the relation R defines ethical preferences over U_n if and only if the relation R' defines ethical preferences over $\mathcal{S}$.

Proof:

- i) R' is welldefined and the reflexivity and transitivity properties are obvious.
- ii) First, we have to prove that R is welldefined. Assume $u, v \in U_n$ and $u = F(X) = F(X')$, $v = F(Y) = F(Y')$ for different social states $X, X' \in \mathcal{S}$ and $Y, Y' \in \mathcal{S}$. However, $F(X) = F(X')$ and $F(Y) = F(Y')$ imply that $X I' X'$ and $Y I' Y'$ so the relation $u R v$ is unique. Furthermore U_n is equal to the range of F according to A3 and R' is defined for all $(u, v) \in U_n \times U_n$. The reflexive and transitive properties follow immediately from R' .
- iii) First assume R to be ethical preferences over U_n . We have to prove R' to be a complete, symmetrical and strictly monotone pre-ordering. The completeness follows from the completeness of R .

Now consider $X, Y \in \mathcal{S}$ and a permutation π of I such that

$(X(i), i) \succeq (Y(\pi(i)), \pi(i)) \forall i \in I$. If $u = F(X)$, $v = F(Y)$ we have

$u(i) \geq v(\pi(i)) \forall i \in I$. The symmetry property of R implies that $v I \pi v$ and the strictly monotone property that $u R \pi v$ and thus $u R v$. This also means that $X P' Y$.

Furthermore if $(X(i), i) \succ (Y(\pi(i)), \pi(i))$ for some $i \in I$ we have $u(i) > v(\pi(i))$ for some $i \in I$ and by that $u P v$ i.e. $X P' Y$. This proves that R' is ethical preferences over $\mathcal{S}$.

Now assume R' to define ethical preferences over $\mathcal{S}$. If $u, v \in U_n$ there always exist $X, Y \in \mathcal{S}$ so that $u = F(X)$ and $v = F(Y)$. This is true because $\mathcal{P}$ is connected and this implies that to each utility level there is always a corresponding position in $\mathcal{P}$.

$u \geq v$ implies that $(X(i), i) \succeq (Y(i), i) \forall i \in I$ and by that $X R' Y$ and $X R Y$. If furthermore $u \neq v$ then $(X(i), i) \succ (Y(i), i)$ for some $i \in I$ and from that

$X P' Y$ and $u P v$.

We also conclude that $u(i) = v(\pi(i))$, for some permutation π of I , implies that $(X(i), i) \sim (Y(\pi(i)), \pi(i))$ i.e. $X I' Y$ and $u I v$. This proves that R defines ethical preferences over U_n . ■

Theorem 2: Under A_1 , A_2 and A_3 T2:3 is not true if U_n contains more than one point.

Proof: We shall prove the existence of subjective preferences $\succeq'$ over $\mathcal{P}$ such that no extension $\succeq$ to $\mathcal{P} \times I$ exists making subjective preferences R over $\mathcal{S}$ corresponding to $\succeq'$ ethical preferences.

Consider a social state X (and individual i) where $X(i) \not\succeq_1 X(j) \forall j \neq i$.

Such states exist because of A_3 and because the utility scale A contains more than one point. Let π be a permutation of I such that $\pi(i) \neq i$. Then we have $X P_i \pi X$ according to the definition of R_i ($X R_i Y \Leftrightarrow x(i) \succeq'_i Y(i)$) but this proves R_i not to define ethical preferences. ■

Ethical preferences in D3:4 have been defined using an ordinal utility scale as a basis. Such a scale, however, is only unique up to monotone transformations. This means that for a given space $\mathcal{S}$ of social states there are many possible definitions of the "same" ethical preferences. Let us look a little closer at this.

For a given continuous representation f of subjective preferences $\succeq$ over $\mathcal{P} \times I$ the function $g \circ f$ represents the same preferences if $g : R \rightarrow R$ is a strictly monotone and increasing function. If we want $g \circ f$ to be continuous we also have to presume g to be continuous. The utility scale A will be transformed by g and $g(A) = \{x : x = g(t), t \in A\}$. The new utility space is $g(U_n) = \{u : u = g(v) \text{ and } v \in U_n\}$. T2 now tells us the unique way of defining ethical preferences R' over $g(U_n)$ if ethical preferences R over U_n are given.

Definition 4: If the relation R defines ethical preferences over U_n and

$g : R \rightarrow R$ is a strictly monotone and continuous transformation then by ethical preferences R' over $g(U_n)$ we mean: $u', v' \in g(U_n)$ and $u' R' v' \Leftrightarrow u' = g(u), v' = g(v)$ and $u R v, g(u)(i) = g(u(i))$ etc.

The definition D4:4 has to be consistent with D2:4 and therefore we prove:

Theorem 3: Ethical preferences according to D4:4 are also ethical preferences according to D2:4 and correspond to the same ethical preferences over $\mathcal{J}$.

Proof: It is obvious that R' in D4:2 is a complete pre-ordering over $g(U_n)$. R' has the strictly monotone property because g is strictly monotone. If π is a permutation of I we have for $u', v' \in g(U_n) : u'(i) = v'(\pi(i)) \forall i \in I \Leftrightarrow g(u(i)) = g(v(\pi(i))) \forall i \in I \Leftrightarrow u(i) = v(\pi(i)) \forall i \in I, u, v \in U_n$. So $u I v$ and by that $u' I' v'$. Thus R' defines ethical preferences according to D2:2. We must also check that R and R' defines the same preferences i.e. correspond to the same preferences R'' over $\mathcal{J}$. This is true because: $X, Y \in \mathcal{J}$ and $X R'' Y \Leftrightarrow u R v, u(i) = f(X(i), i), v(i) = f(Y(i), i) \forall i \in I \Leftrightarrow u' R' v, u'(i) = g(f(X(i), i)), v(i) = g(f(Y(i), i)) \forall i \in I$ and g is the transformation between individual position and utility level. ■

5. Ethical Preferences Defined over the Utility Space with a Variable Number of Individuals

Ethical preferences have been defined (D5:3) over the set $\mathcal{J}$ of social states and (D3:4) over the set U_n of utility distributions among n individuals. These two definitions are equivalent in a certain sense, which is mentioned in sec. 4. Characteristic for the two definitions is that comparisons between different social states, or corresponding utility distributions, with the same and finite number of individuals, are always made. In many cases, however, it is necessary to compare social states of a different number of individuals, and because of that we shall consider a set of utility distributions of different sizes, and over that set define ethical preferences which have to coincide with the old ones in relevant cases. We first define a set

of distributions.

Definition 1: The utility space U is the set of real measurable functions $u : I \rightarrow A$ where A is the utility scale from the previous section and I is an individual set. I may be finite or infinite, but in both cases I is a measure space with a positive measure ν with $0 < \nu(I) < \infty$. In the finite case $\nu(G) = \#G$, where $\#$ denotes the number of individuals in G . Furthermore, $U_I \subset U$ is assumed to be those elements in U which has I as a common definition set.

The set U can be seen as the union of the previously used utility spaces U_n and utility spaces including an infinite number of individuals. We have

$$U = \bigcup_I U_I \text{ and } U_I = U_n \text{ if } I \text{ has } n \text{ individuals.}$$

For a definition of ethical preferences over U we need some principle of how to compare utility distributions including different numbers of individuals. Earlier ethical preferences have been interpreted as those, an individual would have if he did not know his position in a social state. With this in mind it is natural to consider only the relative utility distribution in a social state.

This is also sufficient to get an extension of ethical preferences from U_n to U . We therefore define the space V of relative distributions as a set of increasing functions.

Definition 2: The space V of relative utility distributions is the set of increasing real functions $x : [0,1] \rightarrow A$, where A is the utility scale.

We can also define a function T from U to V which assign to each utility distribution $u \in U$ the relative utility distribution $T(u) = x \in V$.

Definition 3: The connection between elements in U and V is given by a function $T : U \rightarrow V$ according to : $T(u) = x$ for $u \in U$, $x \in V$ where $x(t) = \sup \{a : \frac{\nu\{i:u(i) \leq a, i \in I\}}{\nu(I)} \leq t\}$.

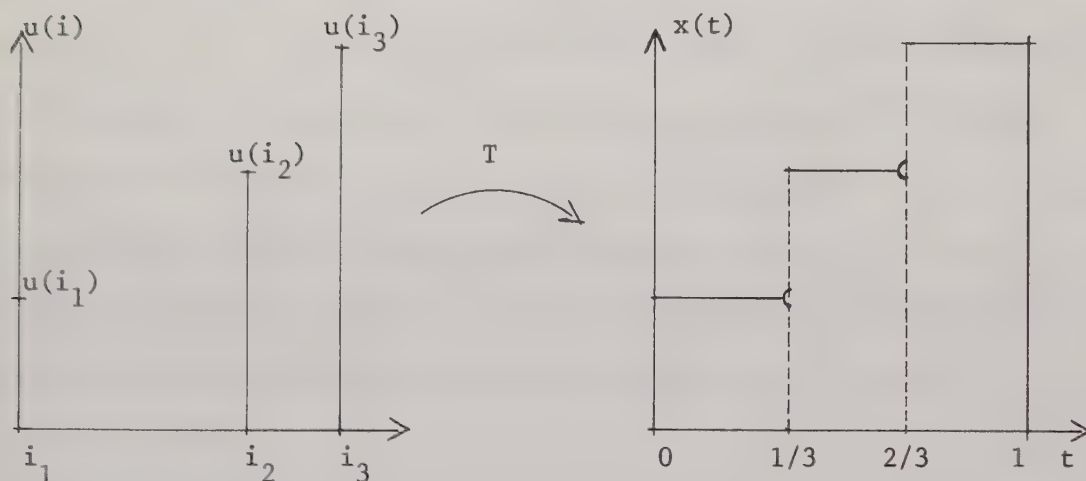
Thus $x(t)$ is the "maximal" utility level, such that the part of the individuals with a utility level $u(i)$, not greater than a , is less or equal to t . It is immediately clear that $x(t)$ is an element in V according to this definition. Before defining ethical preferences we shall analyse (ex., T1-T8) the properties of the transformation T . First consider an example.

Example: Consider a distribution $u \in U_3$ with $u(i_k) = k$, $k = 1, 2, 3$. The corresponding $x = T(u)$ is defined according to:

$$x(t) = 1 \quad \text{for } 0 \leq t < 1/3$$

$$x(t) = 2 \quad \text{for } 1/3 \leq t < 2/3$$

$$x(t) = 3 \quad \text{for } 2/3 \leq t \leq 1.$$



In general an element in U_n is mapped by T on a step function in V . We state the exact form of this as a formal theorem.

Theorem 1: If $u \in U_n$ then $T(u)$ is an increasing stepfunction such that $T(u)(t) = a_k$ for $t \in \left[\frac{k-1}{n}, \frac{k}{n} \right)$ $k = 1, 2, \dots, n$ and $T(u)(1) = a_n$. The sequence $\{a_k\}_{k=1}^n$ consists of the utility levels in u arranged in an increasing sequence.

To prove this theorem it is sufficient to consider once more the preceding example. From the definition of T follows also immediately.

Theorem 2: $u, u' \in U_I$ and $u \geq u' \Rightarrow T(u) \geq T(u')$.

For ν -integrable distributions in U we can also prove that the integral of a distribution is the same as the integral of the corresponding relative distribution. The aim of the next theorems is to prove this. To simplify the notation we assume $\nu(I) = 1$. The results, however, are true also for $\nu(I) \neq 1$.

We shall use the following notation: For ν -integrable $u \in U$ define $G_k = \{i : \frac{k}{n} \leq u(i) < \frac{k+1}{n}, i \in I\}$ and $E_k = \bigcup_{j \leq k} G_j$, $k = 0, \pm 1, \pm 2, \dots$

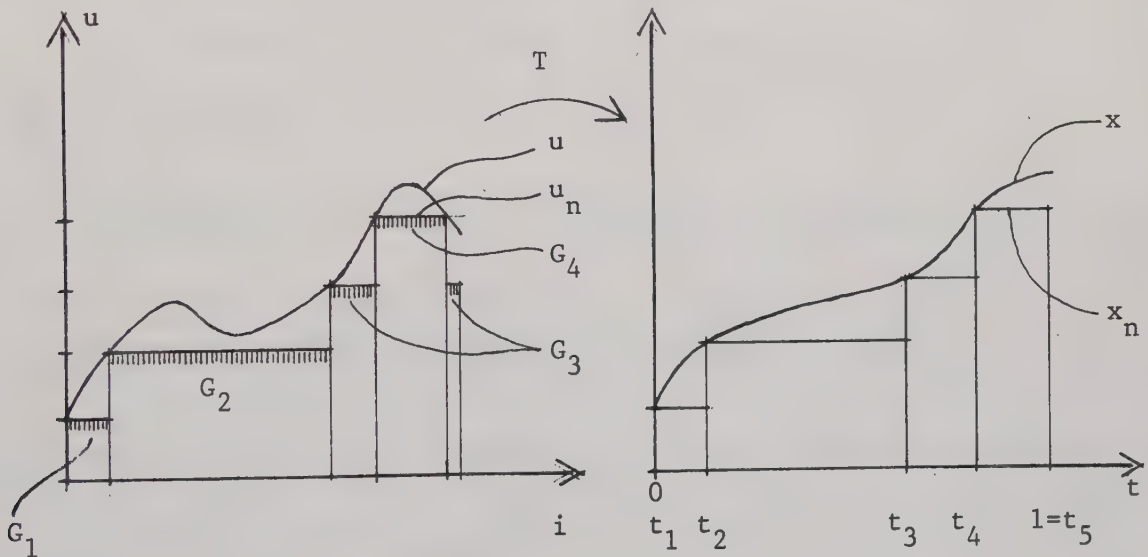
Furthermore $u_n(i) = \frac{k}{n}$ for $i \in G_k$ and $t_k = \nu(E_{k-1})$.

Thus $\nu\{i : u(i) < \frac{k}{n}, i \in I\} = t_k$.

We also define $x_n(t) = \frac{k}{n}$ for $t_k \leq t < t_{k+1}$, $k = 0, \pm 1, \pm 2, \dots$

and $T(u) = x$.

The definitions can be illustrated in a figure.



A common way of defining $\int_I u d\nu$ is to approximate u with step functions u_n and put $\int_I u d\nu = \lim_{n \rightarrow \infty} \int_I u_n d\nu$. It is also true that $T(u_n) = x_n$ and this can be proved in the same simple way as T1. We shall use this result without further comments. We can also prove that x_n is an approximation of x .

Theorem 3: $x_n \leq x$ and $|x(t) - x_n(t)| \leq \frac{1}{n}$ a.e.

Proof: We first prove that $x(t_k) = \frac{k}{n}$ for all k .

Consider $x(t) = \sup \{a : v\{i : u(i) \leq a\} \leq t\}$ for $t = t_k$.

1) $a = \frac{k}{n}$ implies that $v\{i : u(i) \leq \frac{k}{n}\} \geq t_k$ and thus $x(t_k) \geq \frac{k}{n}$.

2) $a = \frac{k}{n} - \varepsilon$, $\varepsilon > 0$ implies that $v\{i : u(i) \leq \frac{k}{n} - \varepsilon\} \leq v\{i : u(i) < \frac{k}{n}\} = t_k$ and thus $x(t_k) \geq \frac{k}{n}$.

1) and 2) $\Rightarrow x(t_k) = \frac{k}{n}$.

We now know that $x(t_k) = x_n(t_k) = \frac{k}{n}$. Because x is an increasing function and x_n is constant in (t_k, t_{k+1}) we also have $x_n \leq x$. Furthermore the property $x(t_k) = x_n(t_k) = \frac{k}{n}$ implies that $|x(t) - x_n(t)| < \frac{1}{n}$ for $t \in (0, 1)$ i.e. a.e. in $[0, 1]$. ■

Theorem 4: $\int_I u_n dv = \int_0^1 x_n(t) dt$

Proof: $\int_I u_n dv = \sum_{k=-\infty}^{+\infty} \frac{k}{n} \cdot v(G_k) = \sum_{k=-\infty}^{+\infty} \frac{k}{n} (t_{k+1} - t_k) = \int_0^1 x_n(t) dt$. ■

Theorem 5: x is (Lebesgue-) integrable.

Proof: x is increasing and therefore also measurable. Furthermore

$|x - x_n| \leq \frac{1}{n}$ (T3) and x_n integrable implies that x is integrable in $[0, 1]$. ■

Finally, we can also prove that integration is invariant under the transformation T .

Theorem 6: For integrable $u \in U$ $T(u)$ is integrable and: $\int_I^1 u dv = \int_0^1 T(u) dt$.

Proof: The theorem is true for $u_n \in U$ according to T4 and in general we have:

$$\begin{aligned} & \left| \int_I u dv - \int_0^1 T(u) dt \right| = \left| \int_I u dv - \int_0^1 x dt \right| = \\ & = \left| \int_I u dv - \int_0^1 x_n dt + \int_0^1 (x_n - x) dt \right| = \left| \int_I u dv - \int_I u_n dv + \right. \\ & \left. + \int_0^1 (x_n - x) dt \right| \leq \int_I |u - u_n| dv + \int_0^1 |x_n - x| dt \rightarrow 0 \end{aligned}$$

when $n \rightarrow \infty$, i.e. $\int_I u dv = \int_0^1 T(u) dt$. $T(u)$ is integrable according to T5. ■

1 If $v(I) \neq 1$ we have instead $\int_I u dv = v(I) \int_0^1 T(u) dt$.

The reverse is also true i.e. that integrability by $T(u)$ implies integrability by $u \in U$.

Theorem 7: $x = T(u)$, $u \in U$ integrable implies that u is integrable.

Proof: T is defined by $x = T(u)$ and

$$x(t) = \sup \{a : \forall i : u(i) \leq a, i \in I\} \leq t\}.$$

From this definition follows directly that $x^+ = T(u^+)$ and $x^- = T(u^-)$ where $x^+(t) = \max(t, 0)$ etc.

However, $x = x^+ - x^-$ integrable implies that x^+ and x^- integrable and furthermore $x^+, x^- \in V$.

We can now use this to prove that u^+ and u^- , and thereby $u = u^+ - u^-$, are integrable.

First define $\Psi_n : R \rightarrow R$ according to $\Psi_n(t) = \min(t, n)$. The properties of Ψ_n are: $\Psi_n \circ x \in V$, $\Psi_n \circ u \in U$ for $x \in V$ and $u \in U$. Furthermore, from the definition of T follows that $T(\Psi_n \circ u) = \Psi_n \circ T(u)$.

The integrability of u^+ (and in the same way the integrability of u^-) follows from Lebesgue's theorem about monotone convergence and from T6:

$$\begin{aligned} 0 &\leq \Psi_n \circ u^+ \nearrow u^+ \text{ when } n \nearrow \infty \text{ and } \int_I \Psi_n \circ u^+ d\nu = \\ &= \int_0^1 T(\Psi_n \circ u^+) dt = \int_0^1 \Psi_n \circ T(u^+) dt \leq \int_0^1 T(u^+) dt = \int_0^1 T(u)^+ dt = \\ &= \int_0^1 x^+ dt < \infty \text{ if } x \text{ integrable.} \end{aligned}$$

$\Psi_n \circ u^+$ is integrable because u^+ is measurable and converges monotonously to u^+ . The integrals $\int_I \Psi_n \circ u^+ d\nu$ are bounded by $\int_0^1 x^+ dt$ and u^+ is thereby also integrable.

u^+ and u^- integrable implies that $u = u^+ - u^-$ integrable and the proof is complete. ■

Next theorem will be used in chapter III.

Consider a partition of $I = G_1 \cup G_2$, $G_1 \cap G_2 = \emptyset$ where G_1 and G_2 are measurable with measure > 0 . Denote with u_G the restriction of $u \in U_I$ to $G \subset I$.

We can then prove that if the subdistributions u_{G_1} and v_{G_1} , u_{G_2} and v_{G_2} have the same relative distribution then even u and v have this property.

Theorem 8: $u, v \in U_I$ and $T(u_{G_1}) = T(v_{G_1})$, $T(u_{G_2}) = T(v_{G_2})$ imply that $T(u) = T(v)$.

Proof: We shall use the following notation: $v(u, a) = v\{i : u(i) \leq a, i \in I\}$

$v_k(u, a) = v\{i : u(i) \leq a, i \in G_k\}$ $k = 1, 2$.

T is defined according to: $T(u)(t) = \sup \{a : \frac{v(u, a)}{v(I)} \leq t\}$

i) We first prove that $T(u) = T(v)$ for $u, v \in I$ implies that $v(u, a) \geq v(v, a - \varepsilon)$ for all $\varepsilon > 0$ (1).

If $t = \frac{v(u, a)}{v(I)}$ then $T(u)(t) \geq a$ according to the definition of T .

If, however, (1) was not true i.e. $v(u, a) < v(v, a - \varepsilon)$ for some $\varepsilon > 0$ then $\frac{v(v, a - \varepsilon)}{v(I)} > t$ and thereby also $T(v)(t) \leq a - \varepsilon$.

Thus, $T(v)(t) \leq a - \varepsilon < a \leq T(u)(t)$ which is a contradiction. Relation (1) is proved.

ii) Now consider an arbitrary $t \in [0, 1]$ and $a \in \mathbb{R}$ such that $T(u)(t) > a$. In that case we also have: $v(u, a) \leq t$ and $v(u, a) = v_1(u, a) + v_2(u, a) \leq t$.

From (1) follows: $v_1(v, a - \varepsilon) + v_2(v, a - \varepsilon) \leq v_1(u, a) + v_2(u, a) \leq t$ because $T(u_{G_1}) = T(v_{G_1})$ and $T(u_{G_2}) = T(v_{G_2})$.

Thus, $v(v, a - \varepsilon) \leq t$ and thereby $T(v)(t) \geq a - \varepsilon$. We have now proved that $T(u)(t) > a \Rightarrow T(v)(t) \geq a - \varepsilon$. It follows that $T(u)(t) \leq T(v)(t)$ for all $t \in [0, 1]$. On symmetry grounds even $T(v)(t) \leq T(u)(t)$ for $t \in [0, 1]$ must be true and $T(u) = T(v)$. ■

We can now return to the definition of ethical preferences.

Since only relative distributions matters when defining ethical preferences over U , we can make the formal definitions primarily over V instead of over U .

Definition 4: By ethical preferences over V we mean a complete and strictly monotone¹ pre-ordering over V .

Definition 5: By ethical preferences over U or U_I we mean a relation R over U according to $u R v$ for $u, v \in U$ if and only if $T(u) R' T(v)$ where R' defines ethical preferences over V .

We shall prove that the definition of ethical preferences according to D5 is an extension of the earlier definition, but first we have to define the meaning of symmetry in U .

Definition 6: For $u, v \in U_I$ u is a permutation of v , denoted $u = \pi v$, if and only if and $T(u) = T(v)$.

Theorem 9: For $u, v \in U_n$ the definition D6 of a permutation is exactly the same as the usual one.

Proof: For $u, v \in U_n$ we have $T(u)(t) = a_k$ for $\frac{k-1}{n} \leq t < \frac{k}{n}$, $k = 1, 2, \dots, n$ and $T(u)(1) = a_n$. a_k are the utility levels in the distribution u ordered in an increasing sequence. The definition of $T(v)(t)$ is the same. It is now obvious that $T(u) = T(v)$ if and only if the utility levels in u are a permutation of the utility levels in v . ■

Definition 7: A pre-ordering R over U is symmetrical if $u = \pi v$, for a permutation π according to D5, implies that $u I v$ where I is the indifference relation corresponding to R .

Theorem 10: Ethical preferences over U_I is a complete, symmetrical and

¹ A pre-ordering R over V is strictly monotone if $x, y \in V$, $x \geq y$ a.e. and $x \neq y$ in a set with Lebesgue-measure > 0 implies that $x P y$.

strictly monotone pre-ordering over U_I for each individual set I .

Proof: Assume R to define ethical preferences over U_I and R' the corresponding preferences over V . For $u, v \in U_I$ we have $u R v$ if $T(u) R' T(v)$. The reflexivity and transitivity properties of a pre-ordering is obviously transformed from R' to R and so is also the completeness property. To prove the strictly monotone property it is sufficient to prove that T is a strictly monotone transformation.

According to T2 we have: $u \succeq u' \Rightarrow T(u) \succeq T(u')$ for $u, u' \in U_I$ and there only remains to prove the strictly monotone property. This can be proved by showing $\int_0^1 [\varphi_n \circ T(u) - \varphi_n \circ T(u')] dt > 0$ for some n if $u \succeq u'$ and $u(i) > u'(i)$ in a set with measure > 0 . φ_n is defined by: $\varphi_n(t) = t$ if $|t| \leq n$ and $\varphi_n(t) = n$ if $t > n$, $\varphi_n(t) = -n$ if $t < -n$.

For such u and u' , however, we have according to T6:

$\int_0^1 [\varphi_n \circ T(u) - \varphi_n \circ T(u')] dt = \int_I [\varphi_n \circ u - \varphi_n \circ u'] dv > 0$ for some n and the proof is complete. (We have used: $T(\varphi_n \circ u) = \varphi_n \circ T(u)$, compare with Ψ_n in T 7!) ■

T10:5 shows that ethical preferences according to D5:5 are formally exactly the same as ethical preferences according to D3:4, but T9:5 also shows that the symmetry property in U_n defined in sec. 4 coincide with the definition of symmetry in sec. 5 for elements $i \in U \cap U_n$. Furthermore, the completeness and the strictly monotone properties are obviously the same in D3:4 and D4:5 for elements in $U \cap U_n$. Thus, each pre-ordering over U defining ethical preferences according to D4:5 also defines ethical preferences according to D3:4 for elements in $U \cap U_n$ and the similarity between D5:5 and D3:4 of ethical preferences is more than formal, D5:5 is an extension of D3:4.

6. Continuous Ethical Preferences

In economic theory it is often assumed that preferences are continuous. This is in general only a mathematical assumption, and has no real economic inter-

pretation. This is also the case if we had made continuity assumptions about ethical preferences including only finite distributions and a fixed number of individuals. However, if we are interested in comparing distributions of a fixed number of individuals, with distributions of an increasing number of individuals, i.e. asymptotic properties of ethical preferences, there may be real economic contents in a continuity assumption.

Asymptotic properties can be described by properties of preferences over distributions where the individual set I has a continuum of elements.

A problem in this section will be to what extent preferences over infinite distributions will be determined by preferences over finite distributions, and we shall show that under certain continuity assumptions the infinite preferences will be completely determined by the finite ones.

A definition of continuity requires a definition of a topology, and in this section we shall consider not a set of measurable distributions, but a set of integrable distributions. We make the topological definitions, however, in the set of relative distributions. We state this as a formal definition.

Definition 1: U_I is the metric space of integrable utility distributions $u : I \rightarrow A$ where I is a fixed individual set and $d(u, u') = \int_I |u - u'| d\nu$ is the distance between u and u' in U_I . The union of all such sets are denoted $U = \bigcup_I U_I$.

Definition 2: The space of relative utility distributions V is a metric space of Lebesgue-integrable increasing functions $x : [0, 1] \rightarrow A$, where $d(x, x') = \int_0^1 |x - x'| dt$ is the distance between x and x' .

T6 and T7 in sec. 5 shows that V is exactly the corresponding set to U under the transformation T i.e. $T(U) \subset V$ ($T(U)=V$).

The topologies defined on U_I and V also have the property to make the transformation T continuous.

Theorem 1: The restriction of T to U_I is continuous.

Proof: Consider $u, u' \in U$ and define $\underline{u}$ and $\bar{u}$ according to: $\underline{u} = \min(u, u')$ and $\bar{u} = \max(u, u')$. Furthermore $\underline{T(u)} = \min(T(u), T(u'))$ and $\overline{T(u)} = \max(T(u), T(u'))$.

Now follows:

- (1) $\underline{u} \leq u \leq \bar{u}$ and $\underline{u} \leq u' \leq u$ from the definition of $\underline{u}$ and $\bar{u}$.
- (2) $T(\underline{u}) \leq T(u) \leq T(\bar{u})$ and $T(\underline{u}) \leq T(u') \leq T(\bar{u})$ from (1) and the monotonicity of T .
- (3) $\int_I |u - u'| d\nu = \int_I (\bar{u} - \underline{u}) d\nu$ from the definition of $\bar{u}$ and $\underline{u}$.
- (4)
$$\begin{aligned} \int_0^1 |T(u) - T(u')| dt &= \int_0^1 (\overline{T(u)} - \underline{T(u)}) dt \leq \\ &\leq \int_0^1 (T(\bar{u}) - T(\underline{u})) dt = \int_I (\bar{u} - \underline{u}) d\nu = \int_I |u - u'| d\nu \end{aligned}$$

from the monotonicity of T from T6:5 and from (3).

The continuity of T is now proved because according to (4) $d(T(u), T(u')) \leq d(u, u')$ and thus to each $\varepsilon > 0$ there exists $\delta > 0$ such that $d(u, u') < \delta \Rightarrow d(T(u), T(u')) < \varepsilon$. We can put $\varepsilon = \delta$. ■

We shall also use subsets of U and V .

Definition 3: The set of finite distributions $U^* \subset U$ or the finite relative distributions $V^* \subset V$ is the subspace $U^* = \{u : u \text{ is a real function } I \rightarrow \mathbb{R} \text{ where } I \text{ is finite}\}$ of U and the subspace $V^* = \{x : x \in V \text{ and } x = T(u), u \in U^*\}$ of V . T is the transformation from U to V defined in sec. 5.

The property of V , to have L_1 -topology, is not invariant under monotone transformations of the utility scale A . If, for instance, A is bounded, every increasing real function defined in $[0, 1]$ is integrable, but this is not true if $A = [-\infty, \infty]$. This means that if we assume ethical preferences to be continuous in V this assumption has different contents depending on which utility scale A is chosen.

The definition of continuous preferences is the usual one.

Definition 4: A pre-ordering R over a topological space S is continuous if the sets $\{x : x \in S \text{ and } x R y\}$ and $\{x : x \in S \text{ and } y R x\}$ are closed for each $y \in S$.

We also define a strictly monotone pre-ordering.

Definition 5: A pre-ordering R over V or V^* is strictly monotone if $x \geq y$ a.e. implies that $x R y$ and if furthermore $x(t) > y(t)$ for $t \in m$ where m is of measure > 0 then $x P y$ where P is the strict ordering corresponding to R . $x, y \in V$ or $x, y \in V^*$.

The natural definition of continuous ethical preferences is:

Definition 6: By continuous ethical preferences over V we mean a complete, strictly monotone and continuous pre-ordering over V .

The rest of this section we shall use to prove that continuous ethical preferences over V^* , the set of finite relative distributions, in a unique way determine preferences over the whole space V .

This means that if we want to work with an infinite number of individuals, we need not bother about preferences over infinite distributions, because these will follow from the finite case.

The strictly monotone property in D6:6 does not, however, follow from a strictly monotone property in the finite case, so by assuming this property in the infinite case we have a possible and natural way to restrict preferences over finite distributions.

Theorem 2: V is a complete metric space.

Proof: Let $x^n \in V$ be a Cauchy sequence. According to the Riesz-Fischer theorem there exists $x \in L_1(0,1)$ ($L_1(0,1)$ = the set of Lebesgue-integrable functions from $[0,1] \rightarrow \mathbb{R}$) such that $x^n \rightarrow x$ in the L_1 -topology. Then there also exists a subsequence x^{n_k} converging to x a.e. (compare the proof of the

Riesz-Fischer theorem in Hörmander¹). This proves that x is increasing a.e. ■

Theorem 3: V^* is a dense subset of V .

Proof: Each function in L_1 can be approximated by step functions in the L_1 -norm. This is a wellknown theorem in integration theory. ■

Theorem 4: V (and V^*) is a separable metric space.

Proof: We just have to prove the existence of a countable dense subset S of V . Let S be a subset of V^* where $x(t)$ for $x \in S$ only have rational values. S is obviously countable and dense in V^* and thereby also in V . ■

T3 and T4 secure the existence of a continuous ordinal utility function for every complete and continuous pre-ordering R over V (or V^*). We state this as a theorem. The proof can be found in e.g. Debreu (1964).

Theorem 5: Every complete and continuous pre-ordering R over V^* can be represented by a continuous ordinal utility function.

We can now prove that ethical preferences over the set V^* of finite relative utility distributions in a unique way induce a pre-ordering over the set V of infinite utility distributions.

Theorem 6: If the relation R^* defines continuous ethical preferences over V^* then there exists one and only one complete and continuous pre-ordering R over V to which R^* is a subrelation. Furthermore R is monotone.

Proof: We consider V^* as a subspace of V with the induced topology; both are metric spaces with the L_1 -metric.

The space V^* is separable (T3) and by T4 there exists a continuous ordinal utility function $f^*(x) \geq f^*(y) \Leftrightarrow xR^*y$ for $x, y \in V^*$.

¹ Hörmander (1964) page 20.

The function f^* can be extended to a function f defined and continuous in U which coincide with f^* in U^* i.e. $f(x) = f^*(x)$ for $x \in V^*$.¹ This extension is also unique. To prove this we assume that f and g are continuous functions in U and that $f(x) = g(x)$ for $x \in V^*$. If $x \in V$ there exists a sequence $x^n \in V^*$ converging to x , $x^n \rightarrow x$, $n \rightarrow \infty$. Thus, $f(x) = \lim_{n \rightarrow \infty} f(x^n)$ and $g(x) = \lim_{n \rightarrow \infty} g(x^n)$. However, $f(x^n) = g(x^n)$ and thereby $f(x) = g(x)$. We conclude that $f = g$.

Now f defines a complete and continuous pre-ordering R over V :

$$x R y \Leftrightarrow f(x) \geq f(y) \text{ for } x, y \in V.$$

This pre-ordering is unique because every complete continuous pre-ordering over V has a continuous representation f according to T3 and T4, which is a unique extension of the restriction of f to V^* .

It remains to prove that R is monotone. Let $x, y \in V$ and $x \geq y$ a.e. First assume x and y to be bounded i.e. there exists a real number α such that $|x(i)| \leq \alpha$, $|y(i)| \leq \alpha$ a.e. In that case there exists a decreasing sequence $x_n \in V^*$, $x_n \searrow x$ converging to x and an increasing sequence $y_n \in V^*$, $y_n \nearrow y$ converging to y . Thus, $x_n \geq y_n$ and $x_n R y_n$. Continuity implies that $x R y$.

If x and y are not bounded we define $x_n(i) = x(i)$ if $|x(i)| < n$ and $x_n(i) = n$ for the remaining points. y_n is defined in the same way. We now have: $x_n \rightarrow x$, $y_n \rightarrow y$ and $x_n \geq y_n$ and therefore $x_n R y_n$ and from the continuity: $x R y$. ■

In T6 we have proved that the monotone property of the pre-ordering R^* over V^* implies a monotone property of R over V . R will not, however, necessarily be strictly monotone even if R^* has this property. A utility function can be given, which represents a strictly monotone pre-ordering over V^* but not over V . We conclude this section by proving the existence of such a function.

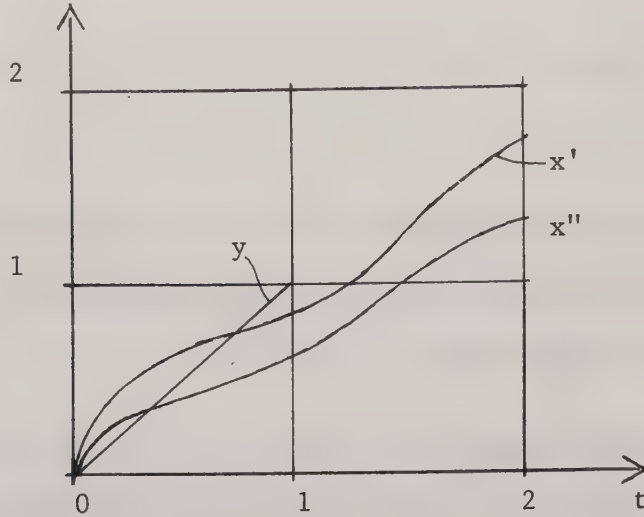
¹ See e.g. problem 2 sec. 17 in Simmons (1963).

Assume the utility scale to be $A = [0, 2]$ and consider the real function

$$f(x) = \int_0^1 x \, dt + \left[\int_0^1 (x-y)^+ \, dt \right] \cdot \int_1^2 x \, dt + \frac{1}{4} \left[\int_0^1 (y-x)^+ \, dt \right] \cdot \int_1^2 x \, dt \text{ where } y(t) = t$$

for $0 \leq t \leq 1$ and $(x-y)^+ = \max[(y-x), 0]$. $(y-x)^+ = \max[(y-x)^+, 0]$.

This function is continuous in the space V and constant if $x(t) = y(t)$ for $0 \leq t \leq 1$. Thus, this function can not be strictly monotone in all the space V but we can prove that f is monotone in V and strictly monotone in V^* .



We first prove that f is monotone. Let $x' \geq x''$.

$$\begin{aligned} f(x') - f(x'') &= \int_0^1 (x' - x'') \, dt + \left[\int_0^1 (x'' - y)^+ \, dt \right] \cdot \int_1^2 x'' \, dt - \left[\int_0^1 (x' - y)^+ \, dt \right] \cdot \int_1^2 x' \, dt \\ &+ \frac{1}{4} \left[\int_0^1 (y - x')^+ \, dt \right] \cdot \int_1^2 x' \, dt - \frac{1}{4} \left[\int_0^1 (y - x'')^+ \, dt \right] \cdot \int_1^2 x'' \, dt = A + B - C + D - E. \end{aligned}$$

We see that $B - C \geq 0$ and $B - C > 0$ if either $(x' - y)^+ > (x'' - y)^+$ in a set $m \subset [0, 1]$ with $v(m) > 0$ or $(x' - y)^+ > 0$ in a set $m \subset [0, 1]$ with $v(m) > 0$ and $x' > x''$ for a set $m' \subset [1, 2]$ with $v(m') > 0$.

Now consider $A + D - E$.

$$\begin{aligned} D - E &\geq \frac{1}{4} \left[\int_0^1 (y - x')^+ \, dt \right] \cdot \int_1^2 x' \, dt - \frac{1}{4} \left[\int_0^1 (y - x'')^+ \, dt \right] \cdot \int_1^2 x' \, dt = \\ &= \frac{1}{2} \frac{\int_1^2 x' \, dt}{2} \left[\int_0^1 (y - x')^+ \, dt - \int_0^1 (y - x'')^+ \, dt \right] \geq \frac{1}{2} \cdot 1 \cdot [\dots] \text{ because of } [\dots] \leq 0. \end{aligned}$$

However, $x' - x'' \geq - (y - x')^+ + (y - x'')^+$ and therefore $A + D - E \geq 0$. We also have $A + D - E > 0$ if $(y - x'')^+ > (y - x')^+$ in a set $m \subset [0, 1]$ with $v(m) > 0$.

It is now easy to check that $x'(t) > x''(t)$ in a set m with $v(m) > 0$ implies that $f(x') > f(x'')$.

7. Ethical Preferences over the Set of Extended Social States

We conclude this chapter by extending the concept of social states. Our first definition of ethical preferences D5:3 was made over the set $\mathcal{S}$ of social states. For technical reasons, however, we have chosen to work in the space U (or V) instead of $\mathcal{S}$.

The correspondence between U and $\mathcal{S}$ has been described in T1:4. The set $\mathcal{S}$, however, is not always the natural starting point to make collective valuations, and we shall therefore generalize this concept.

Consider the set of social states $\mathcal{S} = \{X : X \text{ is a function: } I \rightarrow \mathcal{P}\}$ where I is an individual set defined in D1:5.

Let $g = \{G_j\}$ be a measurable partition of I i.e. $I = \bigcup_j G_j$, $G_j \cap G_k = \emptyset$ if $j \neq k$. There is only a countable number of sets G_j with $v(G_j) > 0$ so we assume $j = 1, 2, 3, \dots$. We also put $\mathcal{G} = \{g : g \text{ is a measurable partition of } I\}$.

A more general form of social states will be $\mathcal{S} \times \mathcal{G}$ i.e. a social state is not only a description of every individual position, but also include a group division $g \in \mathcal{G}$. The reason for this will be clear in the next chapter. Thus, by $(X, g) \in \mathcal{S} \times \mathcal{G}$ we mean a set of functions $X_G : G \rightarrow \mathcal{P}$ one function for each $G \in g$. An element in $\mathcal{S} \times \mathcal{G}$ is called an extended social state.

Before defining ethical preferences over $\mathcal{S} \times \mathcal{G}$ we introduce the concept subdistribution.

Definition 1: If G is a measurable subset of I with $v(G) > 0$ and $X \in \mathcal{S}$ is a social state then the function $u_G : G \rightarrow f(X(i), i)$, where $f : \mathcal{P} \times I \rightarrow A$ is the ordinal utility function of the considered individual, is a subdistribution to the utility distribution corresponding to X .

We are now ready for a definition of ethical preferences over $\mathcal{S} \times \mathcal{G}$.

Definition 2: By ethical preferences (of individual i) over $\mathcal{S} \times \mathcal{G}$ we mean a relation R according to (X, g) , $(X', g') \in \mathcal{S} \times \mathcal{G}$ and $(X, g) R (X', g')$ if $i \in G \in g$, $i \in G' \in g'$ and $u_G R u'_G$ where R defines the ethical preferences over U of individual i and u_G , u'_G are subdistributions which come from X, G and X', G' .

According to D2:5 an individual ranks social states only according to utility distributions in the group to which he belongs.

It is also possible to restrict groupdivisions to $\mathcal{G}' \subset \mathcal{G}$.

If $g = \left\{ \{I\} \right\}_{i \in I}$ for $g \in \mathcal{G}'$ (I finite) ethical preferences according to D2:5 are exactly the same as subjective preferences and if $g = \{I\}$ for $g \in \mathcal{G}'$ then we are back to the case from section 3.

Ethical preferences according to D2:7 can also be interpreted as the preferences an individual would have if he knew that he was going to occupy some position in X_G in the social state X , but did not know which one, and the probability of getting just anyone would be the same.

Chapter III. FAIR DISTRIBUTIONS

In this chapter we will discuss a collective concept of social justice, or more exact what shall be meant by fair utility distributions in a society. In chapter II the individual concept ethical preferences was defined and we shall now amalgamate these individual concepts to a collective one.

Ethical preferences are a complete ranking of social states, but in the case of social preferences we are primarily interested in the top of a social ranking, which we define as a fair state.

In sec. 8 we shall discuss the general form of a criterion of justice by considering some examples where the problem of fair division is analysed.

One conclusion which can be drawn is, that the exact form of the criterion of justice used, is dependent on what cake is to be divided.

The meaning of different social cakes are therefore discussed in sec. 9.

Using sec. 8 and sec. 9 as a starting point, a definition of distributive justice, a fair distribution or a fair social state, is given in sec. 10. We shall also briefly discuss some possible ways of defining social rankings out of ethical preferences.

The rest of the chapter is used to analyse in formal models the existence of fair states, sec. 13, sec. 14 and to give an interpretation of the concept of justice defined in sec. 10.

8. The Form of a Criterion of Distributive Justice

Consider the following formal model which describes a distribution situation. There exists set I of individuals and each individual can be associated with an element in a set $\mathcal{P}$. A function $X : I \rightarrow \mathcal{P}$ will be called a distribution and by F we denote the set of feasible distributions. We also say that $\{X(i)\}_{i \in I}$

is a division of a cake, which is defined by F .

We shall discuss what will be meant by fair distributions. This claims some kind of measure of the "pieces" $X(i)$, and some kind of criterion to determine which distributions are fair.

These two matters will be discussed in some examples.

First consider a very simple case, a homogeneous and infinitely divisible cake, to be divided among a finite number of individuals. In that case $F = \{X : \sum_{i \in I} X(i) \leq \alpha, X(i) \geq 0\}$ is the feasible set of divisions, $X(i)$ is an additive measure on the pieces of the cake and α is the total amount available.

In this case a fair distribution in general is defined so as to fulfil the condition: $X(i) = X(j)$ for all $i, j \in I$ i.e. everybody has received an equal share.

For the distribution to be good we also claim that $\sum X(i) = \alpha$, i.e. that the whole cake has been distributed. Of course, this presupposes that a greater piece is more desirable than a smaller one. This last property of the distribution is an efficiency property.

In more complex cases; a case where, the cake is not homogeneous and not totally divisible and where different individuals may value different pieces differently and so on, it is not so easy to choose the criterion of justice. A general way to construct new criteria is to formulate the equal-share-principle in an alternative way, which can be used in more complex cases.

Now consider an example where $F = \{X : \bigcup_{i \in I} X(i) = C, X(i) \text{ measurable and } X(i) \cap X(j) = \emptyset \text{ if } i \neq j\}$ for some measure space C , i.e. feasible distributions consist of the set of measurable partitions of a set C . Assume furthermore that to each individual $i \in I$ can be associated a probability measure ν_i with the property $\sum_{j \in I} \nu_i(X(j)) = 1$ for $X \in F$. We can then define a prin-

ciple of justice according to: $X \in F$ is fair if $v_i(X(i)) \geq \frac{1}{n}$ for all $i \in I$ if there are n individuals in I . In that case each individual has got his fair share according to his own taste.

This division problem has been analysed in Dubins and Spanier (1961).

The criterion used in this example will also give an equal share to everyone in our first and simple case, where all tastes were the same. Thus we have an extension of the principle that everyone should have an identical piece.

The main element of justice is that each individual has in a certain sense been treated equally or symmetrically, the rule $v_i(X(i)) \geq \frac{1}{n}$ is the same for all $i \in I$.

One disadvantage of this rule compared to the previous one is the lack of uniqueness. There may be several distributions satisfying the criterion.

The assumption of an additive measure v_i , a cardinal measure, is in many cases too strong, and we shall therefore consider examples where the measure is ordinal.

In his "A Theory of Justice" Rawls¹ has used a maximin principle as a criterion of justice. The existence of an ordinal utility function (U), which is the same for all individuals, is assumed. A social state $\bar{X}$ is fair according to the maximin principle if $\min_{i \in I} U(\bar{X}(i)) \geq \min_{i \in I} U(X(i))$ for all $X \in F$ i.e. if the utility level of the worst-off individual in a social state is maximized.

In the case of an infinite divisible homogeneous cake the maximin principle will recommend an equal-share division as fair. We also see that the maximin principle treats everyone equally, it talks about the worst-off individual, but anyone can be worst-off.

¹ Rawls (1972).

We will get a still more general case if assume the existence of several ordinal utility scales, for instance one for each individual.

This is the normal case in economic micro theory. Thus, consider the division of a commodity bundle.

If $X \in F$ we normally use the word allocation for X , and $X(i)$ is the commodity bundle of individual i . It is assumed that each individual $i \in I$ has ordinal preferences $\succsim_i$ over the set of commodity bundles. The set F is for instance, defined according to: $F = \{X : \sum_{i \in I} X(i) \leq \omega, X(i) \geq 0, X(i) \in R^m, \omega \in R^m\}$. m is the number of commodities and ω is the total amount available of the different commodities.

In this case the relevant measure is not the "physical" measure $X(i)$, but the utility corresponding to the pieces $X(i)$. Even if there exist divisions where $X(i)$ is identical for all individuals, those may not be the "best" divisions because the "utility cake" has not been completely used. We can then define the "fairness-criterion" according to:

$X \in F$ is fair if:

- 1) $X(i) \succsim_i X(j) \forall i, j \in I$
- 2) $\nexists Y \in F$ such that $Y(i) \succsim_i X(i) \forall i, j \in I$
and $Y(i) \succ_i X(i)$ for some $i \in I$.

1) is a claim of freedom from envy in the allocation and 2) is a claim of Pareto-optimality i.e. an efficiency property.

If the fairness criterion is used on our first simple homogeneous cake we will find that 1) implies that everybody gets an equal share and 2) that the whole cake is distributed. Thus, the fairness criterion is also an extension of the equal-share-principle.

It can be proved, see Varian (1974), that under certain conditions there will always exist fair distributions. Furthermore, these will not in general be unique.

Because of the lack of uniqueness we can ask which of two distributions, satisfying the fairness criterion is "most" fair, i.e. if it is possible to strengthen the criterion to get a smaller set of fair distributions. An example of this is the concept coalition fairness.¹ In the model with commodities and with individual preferences over commodity bundles, it is possible to define preferences of groups over commodity bundles. If we apply the fairness criterion to groups instead of to individuals we have coalition fairness, which is a stronger concept than, and even include, the usual fairness concept. All this is proved in Varian (1974). The coalition fairness concept is very useful in large economics, economics including a continuum of individuals. It can be proved that in a large economy the only coalition fair allocations are those, which can be characterized as a competitive equilibrium with equal incomes, i.e. each individual has been assigned exactly the same choice set.

In certain economies there exist neither coalition fair nor fair allocations. This is often the case when production is included in the economic model. The non existence of fair allocations depends on the existence of so called personal goods,² i.e. goods which cannot be transferred from one individual to another e.g. good or bad health, different personal abilities and so on.

These questions are discussed e.g. in Varian (1974) and in Daniel (1975). A main problem is that there can be mutual envy between two individuals, which can not be removed because of the existence of personal goods.

In Varian two types of criteria are used to solve these problems. First, there is a weakening of the fairness criterion in such a way, that one individual can envy another individual's commodity bundle, only if he is able to produce the same commodity bundle. It is, however, the usual fairness criterion used in a slightly different way.

1 See Varian (1974).

2 See Daniel (1975).

The second type of criterion is an extension of coalition fairness in a large economy. Fairness in that case was equivalent to an equal income equilibrium.

Under certain assumptions it is always possible to associate equilibrium prices with a Pareto-optimal allocation and the income of an individual is defined as the value of the commodity bundle held by the individual. An income-fair allocation is then an allocation where all incomes are equal.

The two definitions of a fair distribution have quite different properties and we shall discuss that later on (sec. 9).

In Daniel (1975) the problem of unremovable mutual envy is solved in another way. Daniel defines the concept just allocation, and the main idea is that envy is permitted if it occurs in a symmetrical way. An allocation is just if it is Pareto-optimal, and if the number of individuals who envy a particular individual is equal to the number of individuals that he envies.

It is easy to see that every fair allocation according to the fairness criterion is also a just allocation, but the reverse is not always true.

What conclusions can now be drawn about the meaning of a fair distribution? Some properties are common for all the criteria used in the previous examples.

1) Some kind of efficiency claim is almost always put on a fair distribution.

The Pareto-principle is natural.

2) If a criterion is used to decide if a distribution is fair, then each individual is treated equally or symmetrically under the criterion. We say that the criterion has an equity property.

3) When applied to a simple homogeneous cake the criterion will be equivalent to the "equal-share" principle.

The differences between the criteria often depend on to which cake the criterion is applied. The measure on the pieces of the cake may be unique or depend on the individuals. The measure can be ordinal or cardinal. These dif-

ferences also create different possibilities to formulate equity principles.

Also depending on the used model, it is possible to formulate more or less demanding criteria (fairness, coalition fairness). In general there is, however, no internal inconsistency between the different criteria.

Different opinions about the fairness of a distribution often exist because the criteria used have been applied to different cakes. In Varian (1974), for example, the concepts wealth-fairness and income-fairness are applied to the same economic model and as fair distributions give quite different allocations. In the case of wealth-fair distributions, however, no correction for personal abilities is made, i.e. personal abilities are outside the division problem. In the case of an income-fair distribution those personal goods are in a certain sense also divided so that the cake to share will be different.

The problem of "what is the relevant cake to divide" is the subject of next section.

9. Some Alternative Social Cakes

To illuminate the concept "cake" and the consequences of considering different cakes we shall discuss these matters in a couple of examples.

First consider an exchange economy. There exist m commodities and a commodity bundle is an element in R_+^m . $\omega^i \in R_+^m$ are the initial resources held by individual $i \in I$ and an allocation X is a function $X : I \rightarrow R_+^m$. Feasible allocations are $F = \{X : \sum_{i \in I} X(i) \leq \sum_{i \in I} \omega^i\}$. We also assume that the individual set I is finite, has n members. Each individual $i \in I$ has preferences $\succsim_i$ (a complete pre-ordering) over the set of commodity bundles.

We also define the net trade $Z(i)$ of individual i according to: $Z(i) = X(i) - \omega^i$, where X is an allocation.

Preferences over commodity bundles induce preferences $\succsim'_i$ over net trades

according to: $Z(i) \succeq_i' Z'(i) \Leftrightarrow X(i) \succeq_i X'(i)$

The two types of preferences creates two utility cakes, which can be divided. Depending on whether we want to share all commodities fairly or share merely the benefits of trading, we can use the fairness criterion on allocations X or on net allocations Z .

The two types of fairness are:

- 1) X is a fair allocation if $X(i) \succeq_{i_1} X(j) \forall i, j \in I$ and X Pareto-optimal.
- 2) Z is a fair net trade allocation¹ if $Z(i) \succeq_i' Z(j) \forall i, j \in I$ and Z Pareto-optimal.

In the case of fair allocation the individual has no prior right over his initial bundle ω^i , but in the case of fair net trades he controls ω^i completely and the only things to share are the benefits of trading.

This problem of what is to be included in the cake to share, is most relevant in a theory of social justice. The initial bundle ω^i can, for example, be personal goods, and then we have to decide if these, or which of these, goods are to be included in the common pool of goods which is to be divided. If a person is a very able scientist, to what extent shall he benefit from this kind of personal goods?

The problem of personal goods is handled in different ways in Varian (1974) and in Daniel (1975). They are both considering an economy with production and Varian defines wealth-fair allocations and Daniel just allocations. These two concepts include quite different elements in the cake to be divided.

An allocation is wealth-fair if it is Pareto-optimal and free from envy. This is formally the same as a fair allocation, but the definition of envy will need to be precised. A commodity bundle now consists of two parts, the con-

¹ This concept is due to Schmeidler and Vind (1972).

sumption goods received and the production of certain goods. An individual i is allowed to envy another individual j 's commodity bundle only if i prefers to consume and to produce the bundle of j to the bundle of i .

This means that it is very difficult for an unable person to envy a very able person because the unable one has to do so much more work to produce the same as the able one.

In the extreme case a totally handicapped person can never envy an able person.

Personal goods are not included in the cake to be divided if we use wealth-fair allocation as the aim of distributive justice, because no correction for different abilities is made. It is very much like the definition of fair net trades; the cake to divide is the benefits from cooperation.

Now consider a just allocation. According to Daniel an allocation is just if it is Pareto-optimal and the number of individuals who envy one individual is equal to the number that he envies.

A fair allocation i.e. an allocation where no envy exists, is of course also a just allocation, but the reverse is not always true.

Just as in the case of wealth-fair allocation it is the definition of envy that determines what the cake to divide will include.

Daniel's envy concept include the possibility to envy an individual for his personal abilities. To make this point clearer consider two individuals, one consumption commodity and the two individual's working capacities (to produce the consumption commodity). The time needed for the two individuals to produce one unit of the commodity is different, one of the individuals is more able than the other. If they produce the same amount of the commodity and keep it to themselves, the less able individual will envy the more able individual because the less able has to work harder for the same production

than the more able. Envy is not balanced. The claim on a just allocation is that no envy exists, or that both individuals envy each other, so in this case the able individual has to give some of his production to the less able individual.

We see that in a just allocation there is a correction for individual properties to some extent, i.e. personal goods are included in the cake to be divided. There is no risk that a totally handicapped individual shall starve to death in a just allocation.

There is still, however, the question of how much of the handicap ought to be included in the cake. Is it only lack of production capacity, or is it also the eventual psychical sufferings of the handicap, that we shall take into account? There seems not to be a natural limit unless everything is included in the social cake.

More examples of how different definitions of distributive justice include more or less commodities or personal goods in the social cake can be given. I think it is, however, quite obvious that the definition of a social cake is as fundamental as the definition of the criterion of distributive justice. Bearing this in mind we shall now discuss social justice.

10. Distributive Justice, a Definition

In view of what has been said in sec. 8 and 9 a discussion of distributive justice in a society claims definitions of both the social cake to be divided and the criterion of justice to be used, and these two concepts are not independent of each other. We begin with a precision of the social cake.

In sec. 8 and 9 we have seen examples of different kinds of social cakes and a conclusion which can be drawn is that the degree of difficulty increases rapidly when the cake becomes more complex, includes personal goods, feasible distributions depend on the production in the economy and so on.

In principle the social cake can be of any size but the common starting point, which for instance can be found in Rawls (1972), is to make the cake as large as possible. This means that the common pool of "commodities" shall contain not only physical commodities but also personal properties as the capacity of doing different types of work, handicap, the physical and psychological sufferings of an illness and so on. In other words, everything that is relevant for determining an individual's utility level will be found in our social cake. By considering different social cakes we could talk about different degrees of solidarity in the society. We are, however, going to consider the case of "the most solidarity".

If we can define a fair division of such a cake then we have a theory of how all kinds of goods, work and services, i.e. those commodities (not personal goods) which can be transferred from one individual to another, ought to be distributed. How much medical care does an individual have a right to, which is the just pension sum and so on?

How to divide such a cake is often discussed in a hypothetical situation, e.g. Rawls's "original position", a situation where no individual is yet "placed in his position" i.e. the individual does not know who he is going to be. The advantage of considering this hypothetical situation for discussions is that the degree of freedom is maximal, nothing has yet been fixed and it is easy to plan for different possible real situations.

Necessary for the outcome of agreements in this hypothetical situation to be just, is that everyone is treated equally. Our problem is to derive what agreement about distributive justice rational individuals will agree on.

It is, of course, impossible to formulate one and only one principle of justice that everybody in the hypothetical situation would agree on, so we can only discuss possible and probable candidates and analyse the consequences for distributive justice of a particular choice.

In the introduction a couple of principles have been discussed. Rawls's maximin principle and the fairness criterion were such examples. We saw that the consequences of applying these criteria to social cakes including personal goods could very well be distributions where everybody had got an "equally bad" share. This distribution should probably not have been chosen in the hypothetical situation.

The argument was that if two individuals were in an equally bad position, and one of them could by decreasing his utility level just a bit increase the other individual's utility level very much, then the equal distribution should not have been chosen in the hypothetical situation. The consequences of equality are not that everyone will get an equal utility level rather that there is an adequate utility distribution.

The reason for the unacceptable utility distributions recommended by the fairness criterion or Rawls's maximin principle is not the criterion of justice itself but rather the preferences used. In the hypothetical situation an individual would use ethical preferences rather than subjective preferences. If, for instance, there was a dictator in the hypothetical situation, but of course with the restriction that he did not know his future position, ethical preferences and not subjective preferences would be the relevant preferences for him to use in ranking feasible social states. The top of such a ranking would be a fair state.

In the hypothetical situation, however, there is no dictator but everyone will have to be taken into account in a symmetrical way. This means that we have to amalgamate in some way the different preferences, a problem which, except for the different kinds of preferences, is the same as the problem of Rawls, Varian, Daniel etc.

We shall now try to formalize these ideas.

The model to be used is defined in chapter II and we repeat the notation.

1. There exists a set I of individuals which can be finite or infinite.
2. $\mathcal{P}$ is the set of positions and $\mathcal{P} \times I$ the set of individual positions i.e. we have assigned a special position to a special individual.
3. A social state is a function $X : I \rightarrow \mathcal{P}$.
4. $\mathcal{S}$ is the set of social states and $\mathcal{S} \times \mathcal{G}$ is the set of extended social states where $\mathcal{G}$ is the set of measurable groupdivisions of I .
5. Each individual $i \in I$ has subjective preferences over $\mathcal{P}$ and extended subjective preferences over $\mathcal{P} \times I$. Over $\mathcal{S}$ or $\mathcal{S} \times \mathcal{G}$ each individual has ethical preferences.

In chapter II we have discussed the relation of ethical preferences to the hypothetical situation and the remaining question is how to apply ethical preferences and get a collective concept of distributive justice.

The method used e.g. by Varian (1974) or by Daniel (1975) is to seek for utility distributions based on subjective preferences, which have certain symmetry and efficiency properties. The only change we shall make is to substitute subjective preferences with ethical preferences.

Definition 1: $(X, g) \in \mathcal{S} \times \mathcal{G}$ is a fair social state if:

- 1) (X, g) is Pareto-optimal in the subset of $\mathcal{S} \times \mathcal{G}$ which is feasible social states.
- 2) If x_G^i , $G \in g$ is the utility distribution of group G in X according to the extended subjective preferences of individual i and R_i defines his ethical preferences then $x_{G_i}^i R_i x_G^i$ for $G_i, G \in g$ and $i \in G_i$.

Condition 2 in the definition requires some comment. The meaning of this condition is that no individual envies someone else in the sense that, if he compares the utility distribution in his own group in the state (X, g) with the utility distributions in other groups in (X, g) , he cannot find any that he strictly prefers.

One may also ask why we have used $\mathcal{S} \times \mathcal{Y}$ instead of $\mathcal{S}$ in the definition. The following discussion will clarify the motive.

If everyone has the same ethical preferences there is no problem in defining a fair state. It will be a state $X \in \mathcal{S}$ such as no other feasible state $Y \in \mathcal{S}$ exists which according to ethical preferences is better than X . However, it is hard to assume that individual ethical preferences are all quite the same.

As a first question we can ask if the individual's ethical preferences should include all positions in a social state or just a few. If the individual values only his own position, we are back to subjective preferences and that is too small a part of the state to consider.

The opposite case is to consider every position in the social state, but an example will show that this is not very suitable either.

Assume that the individual set I is divided into disjoint groups $I = G_1 \cup G_2$, $G \cap G_2 = \emptyset$ and that the ethical preferences within a group are the same, but differ between two groups.

The social feasibility set $\mathcal{S}$ consists of two parts $\mathcal{S} = \mathcal{S}_1 \times \mathcal{S}_2$, where $\mathcal{S}_k$ is the feasibilities of the group G_k , $k = 1, 2$ and if $X = (X_1, X_2)$, $X_k \in \mathcal{S}_k$ the substates X_k can be chosen independently of each other. Now, if the individual ethical preferences include only the utility distribution of the own group, both groups may very well find a social state which, according to their ethical preferences, is better than a given state in which the total utility distribution is valued by the ethical preferences.

This is the reason why we have considered the set $\mathcal{S} \times \mathcal{Y}$ instead of $\mathcal{S}$ as a basis for a definition of social optimum, justice.

In the definition of a fair state we have also required as a necessity Pareto-optimum according to ethical preferences. If a fair state should be really good it ought to be a Pareto-optimum according to subjective pref-

erences. Next theorem will clarify this situation.

Theorem 1: Under the axiom of identity a Pareto-optimal state according to ethical preferences (e.P.O), is also Pareto-optimal according to subjective preferences (s.P.O.)

Proof: Let $\mathcal{Y}' \subset \mathcal{Y}$ be the feasible social states and assume $(X, g) \in \mathcal{Y}' \times \mathcal{Y}$ to be e.P.O but not s.P.O. In that case there is a state $Y \in \mathcal{Y}'$ such that

1. $(Y(i), i) \succeq_i (X(i), i) \forall i \in I$ (a.e. in I)
2. $(Y(i), i) \succ_i (X(i), i)$ for some $i \in I$ (for $i \in m \subset I$ with $v(m) > 0$)

$\succeq_i$ defines the individual subjective preferences over $\mathcal{P} \times I$. The notation in i parenthesis is in case I is infinite.

The axiom of identity implies that $(Y(i), i) \succeq_k (X(i), i) \forall i, k \in I$ (for almost every $k, i \in I$) and $(Y(i), i) \succ_k (X(i), i)$ for some $i \in I$ and $\forall k \in I$ (for $i \in m$ with $v(m) > 0$ and for almost every $k \in I$). Now immediately follows from the monotonicity of ethical preferences that

1. $(Y, g) R_i (X, g) \forall i \in I$ (a.e. in I)
2. $(Y, g) P_i (X, g)$ for some $i \in I$

(for $i \in m \subset I$ with $v(m) > 0$). Hence (X, g) is not e.P.O. which is a contradiction. R_i defines the ethical preferences of individual $i \in I$. ■

11. Social Rankings

Ethical preferences can also be used for defining rankings of social states. There are two natural ways to go, none of them is, however, quite satisfactory. First consider a definition which takes distributional considerations into account.

Definition 1: $(X', g'), (X'', g'') \in \mathcal{Y} \times \mathcal{Y}$ and $(X', g') P^* (X'', g'')$ according to the weak Pareto principle P^* if $(X', \{I\}) P_i (X'', \{I\})$ for all $i \in I$ (a.e. in I). P_i defines the strict ethical preferences of individual $i \in I$.

In the principle P^* the individuals value the whole utility distribution in the social state and not only the part belonging to their group. One social state is preferred to another, when everybody thinks so. Ethical preferences take distributional considerations into account and therefore P^* also has this property.

There is, however, some lack of efficiency in the principle, because the reason for considering $\mathcal{S} \times \mathcal{Y}$ instead of $\mathcal{S}$ when a fair state was defined, is still relevant. This drawback can be relaxed by making the definition over $\mathcal{S} \times \mathcal{Y}$ instead of over $\mathcal{S}$, but the distributional considerations will become weaker.

Definition 2: $(X', g'), (X'', g'') \in \mathcal{S} \times \mathcal{Y}$ and $(X', g') P (X'', g'')$ according to the Pareto principle P if $(X', g') P_i (X'', g'')$ for all $i \in I$ (for a.e. in I).

It is also possible to prove that under the axiom of identity P^* and P include the ordinary Pareto principle, i.e. if X' is Pareto superior to X'' according to the Pareto principle and subjective preferences then

$(X', g') P (X'', g'')$ and $(X', g') P^* (X'', g'')$ for all $g', g'' \in \mathcal{Y}$. The proof of this is exactly the same as the proof of T1:10. However, both P and P^* give a more complete ranking of social states than the Pareto principle with subjective preferences. This is the gain of considering restricted preferences, ethical preferences, instead of subjective preferences.

If we combine P and the definition of a fair state D1:10 we can construct a social ranking which has a fair state as an optimal element. One way of doing this can be found in Gärdenfors (1975).

We shall not, however, deal any more with social rankings, but return to the definition of a fair state.

12. An Interpretation of a Fair Social State

In sec. 10 we have defined a fair social state and now we are going to discuss the definition, and also some consequences of the definition for an economic system generating fair social states. It is, of course, impossible to derive the exact form of such an economy; there may not exist one, or there may be one or several, but the definition of a fair social state suggests some features of an economic system.

Social justice has primarily been defined as the result of an agreement in a hypothetical situation; a situation where each individual sees straight ahead a set of possible positions and does not know which one he is going to hold. This situation is by Rawls (1972) called the original position (o.p.) and we shall also use his term.

In sec. 10 a formal model has been defined as a description of o.p.

Taking o.p. as a starting point for a definition of social justice is not very original, but different authors have different opinions in general about the exact form of the criterion of justice.

However, some kind of assumption about unanimity between the participants in o.p. is always made. In the case of Utilitarian criterion and Rawls's maximin principle one often says that the preferences are those an impartial spectator would have and therefore the participants in o.p. would agree on them.

In our case, however, no such assumption about complete unanimity in o.p. is made, instead we assume everybody in o.p. to accept some kinds of common restrictions on the preferences to be used in o.p. The restrictions are those which define ethical preferences.

There was also a rational reason for making these assumptions in a situation of choice like o.p. The assumption of symmetry in the preferences was made because nobody in o.p. knew his coming position in a social state and there

was an equal chance of getting any position (or the probability of getting a special position was not known to the individuals).

The position taken by us when defining social justice lies somewhere between the position of welfare theory, where the preferences used have only small common properties, and the position of Rawls and the Utilitarians, where the existence of totally common social preferences are assumed.

Our starting point is ethical preferences, and in case of totally equal preferences we are back to e.g. the Utilitarian case or to Rawls's case, depending on the exact form of the preferences.

Because of the possible differences of the ethical preferences of the individuals, we have a problem of aggregation before we can get a social concept of justice; ethical preferences only reflect the individual concept.

By making this aggregation we have to take into consideration the conditions for o.p.; the total equality of the participants, and thereby we also assume everybody to have equal rights in the most extensive form.

In our definition of a fair state we express this condition of equality by assuming as a necessity the envy-freedom of a fair state.

There is also a question of how much the ethical preferences ought to include, the total utility distribution of a social state, or just a part of it. If the individuals of two or more groups agree on restricting the preferences to the utility distribution of the own group this is, of course, rational and even necessary in view of the assumption of equal rights in the most extensive form in o.p. This is the reason for considering group divided social states (X, g) instead of X .

A desire for (ethical-) Pareto-optimum of a fair social state can be motivated in the same way. It is a rational decision in o.p.

O.p. is considered a kind of initial hypothetical situation in which decisions about the institutions in the society are to be made. However, if the "right" decisions are made in o.p., there must be fair utility distributions in the real society and one could afterwards, i.e. when all individuals have lived their lives, check this. If (X, g) is a fair social state then each individual, when "looking back" on his own and on everybody else's lives, finds that he prefers the utility distribution in the group to which he belongs, to the utility distributions of other groups.

The utility distributions also have to be (ethical) Pareto-optimal.

An economic system consists of a set of mechanisms which determine the "final" utility or welfare distribution in the society. There is the wage system, the pension system, the social security system, the system of allocation of commodities and services, the educational system etc.

We shall not now discuss in detail what claims will have to be put upon these subsystems of an economic system so that fair states are generated, only mention some common problems which are connected with the definition of a fair state. Later however (in chapter IV) we shall consider the problem of fair wages exhaustively.

A main difficulty with the definition in sec. 10 of a fair social state is how to identify the ethical preferences of an individual. Without such an identification it seems to be impossible to establish rules for the subsystems to go by in order to generate fair social states. One intention with the group division of a fair state is that individuals of similar preferences shall be put together in the same group, and this is, of course, possible only if ethical preferences, in one way or another, can be identified.

There are two problems of the identification of ethical preferences; it is not certain that an individual is willing to reveal his ethical preferences and it is possible that he is not able to reveal them.

In principle, a simple way to identify ethical preferences would be to ask the individual which welfare distribution he prefers in different situations and let the distribution mechanism in the economy be dependent on his answers. However, we have no guarantee that he will in this way reveal his true ethical preferences. If he has special kinds of personal goods; bad abilities of some kind, which may motivate the society to spend greater resources on him than on the average individual, he may exaggerate his demand for support from the society. The preferences he reveals will lie somewhere between his subjective and his ethical preferences, if he is not quite impartial when answering the questions about his ethical preferences.

To avoid this problem of impartiality one could let an outsider make the distribution decisions, but in that case it will be his ethical preferences that will determine the distribution, rather than the preferences of the relevant person.

A more realistic way of revealing the ethical preferences of a special individual is to try to copy the situation of choice which exists in o.p., i.e. the decisions about distributive matters in a special situation are made by an individual before he is really in the situation.

We can interpret this as a system of insurances. An individual does not know what incidents might occur in the future; if some kind of accident will give him a handicap, if he will get some kind of illness, where the treatment requires much resources, if he will loose his job and so on. In all these situations the ethical preferences decide which welfare distribution is fair, and the individual may by giving up some of his momentary welfare insure himself against the different possible situations. Thereby his personal ethical preferences will determine which welfare distribution is relevant to him in different situations. The important thing is, that when choosing an insurance an individual is making interpersonal comparisons; he hypothetically puts himself in different positions.

A system of personal insurances will intuitively solve the efficiency problem in a fair social state, but not solve the claim of envy freedom. To achieve also this goal we have to claim that the possibilities of choosing different kinds of insurances are equal to everyone.

In an economy where, for instance, commodities (and insurances) are allocated by a price system, and where income is the measure of the choice set of commodities, a "fair" distribution of income is essential.

Exactly equal income is not necessarily fair if the income includes compensation for jobs which are not identical. The differences in income may instead reflect the different hardships of different jobs.

If income has other sources than work, unequal income will have to be motivated in the same way; will have its origin in differences in the individual situation.

A condition for a system with insurances and equal possibilities in choosing them to generate fair social states is, that the individual has complete preferences in such situations of choice. This is not always the case. Preferences may be more or less incomplete or may not exist at all, i.e. the individual is not able to reveal his ethical preferences.

Some examples will clarify the situation.

A newborn child (or an unborn child) has its whole life ahead of it and ought to choose insurances which cover a very great set of situations, where the distribution of resources is relevant.

This is, of course, impossible because preferences are created gradually during the whole life and, what make things even more complicated is that the "final" preferences may depend on what kind of life the individual has had. However, we have to assume some kind of stability of preferences, so that there is a stage of life where ethical preferences will, in principle,

be complete and stable. The practical problem is then to decide to what extent the actual ethical preferences of an individual may be used to determine the distribution rules in the society.

I think that we have to take the position that a newborn child is, (of course), completely incapable of making decisions, but that preferences to an increasing extent may be used to decide distribution matters depending on the age or on the mental development of the individual.

In any case, in all the situations where the ethical preferences of the relevant individual cannot be used to decide distribution matters, we have to look for some kind of approximation, and it seems natural that the ethical preferences of somebody else or of another group of individuals decide the distribution.

Typical situations and distribution questions to be answered are, "How much resources ought to be spent on one special individual for education or for medical care before he is grown up?", "How much resources will one generation give up for another?", and so on.

It is not evident who will have to answer these questions, it may be the parents of a child or it may be the society in some meaning. One may ask what preferences are most like the (hypothetical) ethical preferences of the child.

By using someone else's ethical preferences when deciding the welfare level of an individual in a special situation we lose some (ethical) efficiency. The social state need not be Pareto-optimal according to ethical preferences.

A fair state is supposed to be, besides Pareto-optimal, also free from envy when a group division has been made, and comparisons between the welfare distribution of the own group and of others are being made. This property does not seem to be influenced by the fact that it is not always the own

ethical preferences of the individual that determine his welfare level. If the rules concerning the distribution of welfare are equal to all individuals in the situation where the own ethical preferences of the individual can not decide the welfare level, then there may not be a conflict with the envy free property of a fair social state.

We can now summarize what has been said in the following way.

For an economic system to generate fair social states it is necessary in some way or another to reveal the ethical preferences of an individual.

This is not always possible, and in those cases an approximation may be that the preferences of someone but the relevant individual are used as a basis for distributional decisions.

A means of revealing ethical preferences is the existence of an insurance system in the economy. In all the cases, however, where the individual for some reason can not himself choose an insurance, there must yet exist insurances which someone else, according to his ethical preferences, has to choose for the first individual.

Special claims have to be put on a system of insurances to achieve the properties of envy-freedom and ethical Pareto-optimum of a fair state.

We get envy freedom if everyone has the same possibility to choose an insurance, and the ethical Pareto-optimum property implies that there must be as great a set of insurances as possible to choose from.

Insurances have to be interpreted in a very wide meaning. To a newborn child the educational system e.g. is a part of the insurance system, because the education will determine future consumption possibilities, and therefore in o.p. the child would like to make sure that it has the same possibilities of education independently of e.g. the income of his parents, or of in what country he was born etc.

This means that all mechanisms in the economy which are of importance to the welfare of an individual are part of an insurance system in our meaning.

13. Fair Division of a Cake Including one Homogeneous Commodity and Equal Personal Goods

In this section, and in the next, we shall study the existence of fair social states in a formal model. We shall prove the existence of an "approximate" fair state, i.e. the existence of states where condition 1 and 2 in D1:10 of a fair social state are satisfied by an arbitrary accuracy.

The exact meaning of this will be clear later on.

We shall consider a very simple distribution situation. The social cake to be divided will consist of one totally divisible and homogeneous commodity, e.g. money, and personal goods which all are the same for each individual. This means that it is sufficient to indicate the amount of the commodity assigned to a position to be able to determine the "subjective value" of that position.

We now define an economy $\mathcal{E}_1$ by adding some assumptions to the earlier defined (sec. 3 or sec. 10) model.

Assumption 1: There is an infinite number of individuals, each represented by a point in the unit interval $[0,1]$. I is a measure space with the Lebesgue measure ν .

Assumption 2: A position in $\mathcal{P}$ is described by two coordinates (u,v) where $u \in \mathbb{R}$ denotes the quantity of the single commodity to be distributed and v , which is the same for all positions, denotes the personal goods.

An individual position is $X(i) = (u(i), v) \in \mathcal{P} \times I$ and a social state is the function $X : I \rightarrow \mathcal{P}$.

Feasible distributions of the single commodity is $S = \{u : u \text{ is a bounded}$

real integrable function $I \rightarrow R$ with $\int_I u(i)di \leq \alpha$, $\alpha \in R$ and $|u(i)| \leq L$. L is a fixed positive real number.

The set $\mathcal{S}$ of feasible social states is $\{X : X(i) = (u(i), v), u \in S\}$.

Assumption 3: The subjective preferences $\succsim_i$ over $\mathcal{P} \times I$ are described by ordinal utility functions f_i according to $f_i(X(j)) = u(j)$ for $i, j \in I$.

Assumption 4: The utility space U_I is a metric space of integrable real functions $u : I \rightarrow R$ with the distance function $d(u, u') = \int_0^1 |u(i) - u'(i)| di$ for $u, u' \in U_I$. $U_I \times I$ is a metric space with the distance function $d((u, i), (u', i')) = d(u, u') + d(i, i')$ where $d(i, i') = |i - i'|$.

Assumption 5: Ethical preferences R_I over U_I can be represented by an ordinal utility function $F : U_I \times I \rightarrow R$ which is also continuous in $U_I \times I$.

When we talk about the economy $\mathcal{E}_1$ in the future, we always make the assumptions A1, A2, A3 and A4.

Before stating the main theorem of this section we have to make some definitions.

When defining approximative fairness we shall use a "hypothetical" social state as a comparison norm.

A hypothetical state will be denoted (X, g) i.e. the same notation as for elements in $\mathcal{S} \times \mathcal{G}$. The exact definition is.

Definition 1: (X, g) is a hypothetical social state if each individual $i \in I$ can be associated with a utility distribution $u^i \in U_I$.

Ethical preferences are, of course, also defined over the set of hypothetical social states, the "ethical utility" of such a state is $F(u^i, i)$.

We also have to make the definitions of envy and Pareto-optimum precise.

Definition 2: A hypothetical social state (X, g) is free from envy if

$$F(u^i, i) \geq F(u^j, i) \text{ for a.e. } i, j \in I.$$

Definition 3: A hypothetical social state (X, g) is a Pareto-optimum in the set S of feasible distributions if there does not exist $g \in \mathcal{G}$ and $u \in S$ such that $F(u_{G_i}, i) \geq F(u^i, i)$ for a.e. $i \in I$ and $F(u_{G_i}, i) > F(u^i, i)$ for $i \in m \subset I$ with $v(m) > 0$. $i \in G_i \in g$ and u_{G_i} is a distribution in U_I which, according to ethical preferences, is equally valued as the restriction of u to G_i . (cp lemma 1).

Approximative fairness can now be defined.

Definition 4: An extended social state (X, g) is ε -fair, $\varepsilon > 0$ in the set S of feasible distributions if there exists a fair hypothetical social state (X', g') such that $|F(u^i, i) - F(u_{G_i}, i)| < \varepsilon$ for a.e. $i \in I$. u^i is the utility distribution associated with individual i in (X', g') and u_{G_i} is the corresponding element in (X, g) , $i \in G_i \in g$.

The main theorem of this section is the existence of ε -fair social states.

We state this theorem now, but the proof will need some lemmas.

Theorem 1: In the economy $\mathcal{E}_1$ there exist for each $\varepsilon > 0$ ε -fair social states.

Lemma 1: If $T : U_I \rightarrow V$ maps utility distributions into relative utility distributions, then $T(U_I) = U_I$ and $T(u) = u$ for increasing distributions $u \in U_I$ in $\mathcal{E}_1$.

Proof: $T(U_I) \subset U_I$ because $I = [0, 1]$. From the definition of $T : T(u)(t) = \sup \{a : v\{i : u(i) \leq a, i \in I\} \leq t\}$ follows that $T(u)(t) \geq u(t)$ for increasing u because $a = u(t) - \varepsilon$, $\varepsilon > 0$ implies that $v\{i : u(i) \leq a, i \in I\} \leq t$ and therefore $T(u)(t) \geq u(t) - \varepsilon$ for all $\varepsilon > 0$.

We also have $\int_I u dv = \int_0^1 T(u) dt$ (T6:5) which in $\mathcal{E}_1$ is the same as $\int_0^1 u(i) di = \int_0^1 T(u)(i) di$.

Because $T(u) \geq u$ we now have $\int_0^1 |T(u)-u|dt = 0$ and from that $T(u)(t) = u(t)$ a.e., i.e. $T(u) = u$. ■

U is the utility space and V the space of relative utility distributions according to the definitions in sec. 5. We also denote with U^L and V^L the space of bounded distributions i.e. $u \in U^L$ if $u \in U$ and $|u(i)| \leq L$ and $x \in V^L$ if $x \in V$ and $|x(t)| \leq L$.

Lemma 2: V^L is a compact metric subspace of V .

Proof: Consider the space $E = \{x : x \text{ is an increasing real function } [0,1] \rightarrow \mathbb{R} \text{ and } 0 \leq x(t) \leq 1 \text{ for } t \in [0,1]\}$. E is a topological subspace of V^L for $L > 0$ and homeomorphic to V^L . E and V^L are thus simultaneously compact.

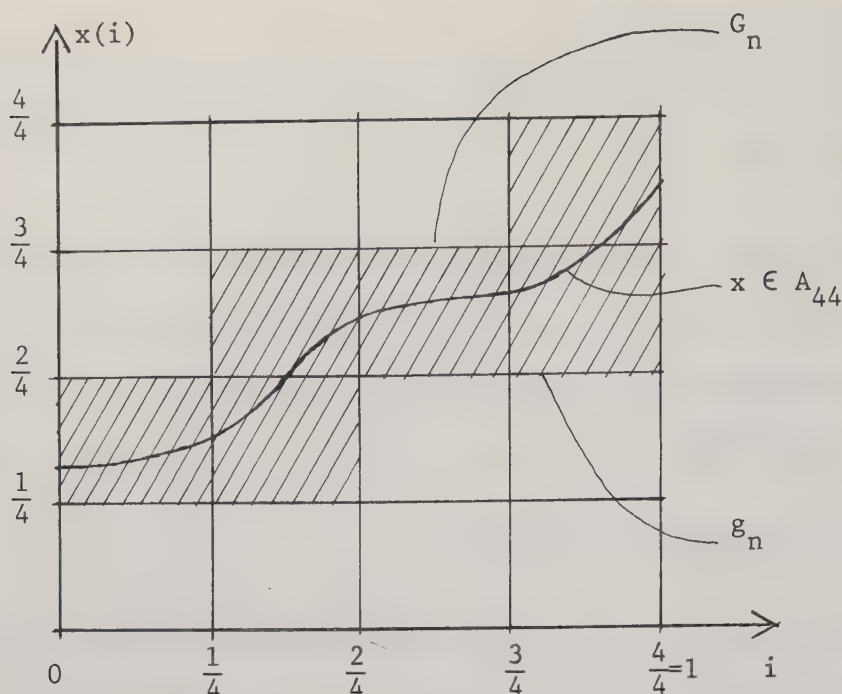
We prove E to be a compact metric space by proving that every sequence contains a convergent subsequence.

Suppose $A = \{x^k\}_{k=1}^{\infty}$ to be an infinite sequence in E , ($x^k \in E$).

Define $I_{nk} = \{i : i \in I \text{ and } \frac{k}{n} \leq i \leq \frac{k+1}{n}\}$ $k = 0, 1, 2, \dots, n$.

To a given n infinite subsets A_{nk} of A are defined in the following way. For some k there must be an infinite number of functions x in A with $x(o) \in I_{nk}$. Let $A_{n0} \subset A$ be a set of such functions.

In the same way $A_{n1} \subset A_{n0}$ is defined to be an infinite set of functions in A_{n0} with $x(\frac{1}{n}) \in I_{nk}$ for some k . $A_{n2} \subset A_{n1}$ is an infinite subset of A_{n1} with $x(\frac{2}{n}) \in I_{nk}$ for some k . By repetition we reach the set A_{nn} .



From the construction of A_{nn} follows the existence of two increasing step-functions G_n and g_n , which are constant in each interval I_{nk} , $k = 0, 1, \dots, n$ and of the property $g_n(i) \leq x(i) \leq G_n(i)$ and $\int_0^1 (G_n(i) - g_n(i)) di \leq \frac{2}{n}$.

We can now choose a set $A_{n+1, n+1} \subset A_{nn}$ in the same way as A_{nn} has been chosen from A and we get step-functions $g_{n+1}(i) \leq x(i) \leq G_{n+1}$, $x \in A_{n+1, n+1}$ and $\int_0^1 (G_{n+1}(i) - g_{n+1}(i)) di \leq \frac{2}{n+1}$.

In this way we obtain two monotone sequences $G_n \searrow$ and $g_n \nearrow$ which also are bounded. Thus, $G(i) = \lim_{k \rightarrow \infty} G_{n+k}(i)$, $G \in E$ and $g(i) = \lim_{k \rightarrow \infty} g_{n+k}(i)$, $g \in E$ exist and $G(i) \geq g(i)$. Furthermore, $G(i) = g(i)$ a.e. because of $\int_0^1 (G_{n+k}(i) - g_{n+k}(i)) di \rightarrow 0$ when $k \rightarrow \infty$.

We have recursively defined sets $A_{n+k, n+k}$ $k = 0, 1, 2, \dots$ of functions and this sequence of sets are completely determined if we e.g. put $n = 2$. ($A = A_{11}$).

We can choose an infinite sequence $\{x^n\}_{n=1}^\infty$ of functions such that $x^n \in A_{nn}$ and $x^n \neq x^k$ if $n \neq k$. In that case we also have $x^n \rightarrow G$ in L_1 -norm. This proves that E and thereby V^L are compact metric spaces. The proof is complete. ■

Now consider the set S and define optimal distributions.

Definition 5: $\bar{u}^i \in S$ is an optimal distribution in S for individual i if

$F(\bar{u}^i, i) \geq F(u, i)$ for all $u \in S$.

The existence of optimal distributions can be proved.

Lemma 3: In $\mathcal{G}_1$ there always exist optimal distributions $\bar{u}^i$ in S for each $i \in I$.

Proof: Consider the set $T(S)$ where T is the transformation from distributions to relative distributions. According to lemma 1 and T6:5 is $T(S) \subset S$.

$T(S) = \{u : u \text{ is an increasing function } [0,1] \rightarrow \mathbb{R}, |u(t)| \leq L \text{ and } \int_0^1 u \, dt \leq \alpha\}$.

$T(S)$ is a closed subset of the compact V^L (lemma 2) and is thereby also compact.

Ethical preferences are represented by a continuous function $F(u, i)$ for fixed i and this, together with the compactness of $T(S)$, implies the existence of $\bar{u}^i \in T(S)$ which maximizes $F(u, i)$ for fixed i in $T(S)$.

Now, $\bar{u}^i$ is an optimal distribution of individual i in S , because $u \in S$ and $F(u, i) > F(\bar{u}^i, i)$ imply that $F(T(u), i) = F(u, i) > F(\bar{u}^i, i)$ and $T(u) \in T(S)$ which is a contradiction to the choice of $\bar{u}^i$. ■

We can now return to T1:

Proof of theorem 1: We first define a fair hypothetical social state (X', g') and then prove the existence of an ε -fair social state (X, g) .

(X', g')

According to lemma 3 there exists for each individual $i \in I$ an optimal distribution $\bar{u}^i \in S$. (X', g') is defined by assigning $\bar{u}^i$ to individual i . (X', g') is a fair hypothetical social state because:

i) From the definition of $\bar{u}^i$ follows directly that (X', g') is free from envy.
 ii) Assume (X', g') not to be a Pareto-optimum relative S . In that case there exist $g \in \mathcal{G}$ and $u \in S$ such that $F(u_{G_i}, i) \geq F(\bar{u}^i, i)$ for a.e. $i \in I$ and

$F(u_{G_i}, i) > F(\bar{u}^i, i)$ for $i \in m \subset I$ with $v(m) > 0$. $i \in G_i \in g$ and u_{G_i} is a distribution in U_I which according to ethical preferences is valued equal to the restriction of u to G_i .

Let u'_{G_i} denote the restriction of u to G_i . According to T6:5 and lemma 1:

$$\frac{1}{v(G_i)} \int_{G_i} u'_{G_i} dt = \int_0^1 u_{G_i} dt \quad (1)$$

$$F(u_{G_i}, i) \geq F(\bar{u}^i, i) \Rightarrow \int_0^1 u_{G_i} dt \geq \alpha \quad (2)$$

$$F(u_{G_i}, i) > F(\bar{u}^i, i) \Rightarrow \int_0^1 u_{G_i} dt > \alpha \quad (3)$$

For some group $G_i \in g$ with measure > 0 relation (3) must hold. Therefore:

$$\int_0^1 u dt = \sum_{G_i \in g} \int_{G_i} u'_{G_i} dt = \sum_{G_i \in g} v(G_i) \cdot \int_0^1 u_{G_i} dt > \sum_{G_i \in g} v(G_i) \cdot \alpha = \alpha.$$

This means that $u \notin S$ which is a contradiction i.e. (X', g') must be a Pareto-optimal hypothetical social state.

The existence of an ε -fair social state

Now consider only increasing distributions in S i.e. the set $T(S)$ where T is the transformation from distributions to relative distributions. According to lemma 1 is $T(u) = u$ for $u \in S$. Therefore, ethical preferences over $T(S)$ can be represented by the same ordinal utility function F as preferences over S .

$T(S)$ is a closed subspace of V^L (e.g. S compact and T continuous (T1:6)) and is therefore, according to lemma 2, compact.

$I = [0, 1]$ is compact and therefore $T(S) \times I$ is also compact.

Now, F is continuous in $T(S) \times I$ (A5) and thereby uniformly continuous. We shall use this:

To each $\varepsilon > 0$ there exists $\delta > 0$ such that

$$d((u, i), (u', i')) < \delta \rightarrow |F(u, i) - F(u', i')| < \varepsilon/3$$

for $(u, i), (u', i') \in T(S) \times I$. (1)

The compactness of $T(S) \times I$ implies the existence of a δ -net in $T(S) \times I$ for every $\delta > 0$ i.e. finite sets $Q \subset T(S)$ and $q \subset I$ such that for $(u,i) \in T(S) \times I$ there always exists $(u',i') \in Q \times q$ with $d((u,i),(u',i')) < \delta$. In that case $d(u,u') < \delta$ and $d(i,i') < \delta$ also is true i.e. Q and q are δ -nets in $T(S)$ and I .

We can now choose a $\delta > 0$ such that for every $(u,i) \in T(S) \times I$ there exists $(u',i) \in Q \times q$ such that $|F(u,i) - F(u',i)| < \varepsilon/3$ where ε and δ are the same as in (1). This also implies that $|F(\bar{u}^i,i) - F(u',i)| < \varepsilon/3$ for some $(u',i) \in Q \times q$. $\bar{u}^i$ is an optimal distribution in $T(S)$ of individual i .

Denote by I' a finite set of individuals $I' = \left\{0, \frac{1}{n}, \frac{2}{n}, \dots, \frac{n}{n}\right\}$ where $\frac{1}{n} < \delta$.

We shall assign a group G_u of individuals to each element $u \in Q$. G_u is defined according to:

- i) Individuals $i \in I'$ are placed in a group G_u where $F(u,i)$ is maximal, $u \in Q$.
- ii) Individuals $i \in I$ with $\frac{k-1}{n} < i < \frac{k}{n}$ are placed in the same group as $i = \frac{k}{n} \in I'$.

We have in this way defined a finite partition of $I = \bigcup_{u \in Q} G_u$, $G_{u'} \cap G_{u''} = \emptyset$ if $u' \neq u''$. Each G_u , $u \in Q$ is the union of a finite number of intervals and is therefore measurable. However, $v(G_u) = 0$ for some $u \in Q$ is possible. We denote by $Q' \subset Q$ those $u \in Q$ where $v(G_u) > 0$ and $g = \{G_u\}_{u \in Q'}$.

We now prove that $|F(\bar{u}^i,i) - F(u,i)| < \varepsilon$ if $i \in G_u$, $u \in Q'$. (2)

The following relations are true for $i \in G_u$, $u \in Q'$ and $i' \in I' \cap G_u$:

- i) $d((u,i'), (u,i)) < \delta$ which also implies that $|F(u,i') - F(u,i)| < \varepsilon/3$ according to (1).

ii) $|F(\bar{u}^{i'},i') - F(\bar{u}^i,i)| < \varepsilon/3$ because the uniform continuity of F implies that $F(\bar{u}^{i'},i') - F(\bar{u}^i,i) \leq F(\bar{u}^{i'},i') - F(\bar{u}^{i'},i) < \varepsilon/3$ if $F(\bar{u}^{i'},i') > F(\bar{u}^i,i)$ and

$$F(\bar{u}^i, i) - F(\bar{u}^{i'}, i') \leq F(\bar{u}^i, i) - F(\bar{u}^i, i') < \varepsilon/3 \text{ if}$$

$$F(\bar{u}^i, i) > F(\bar{u}^{i'}, i') .$$

iii) $|F(\bar{u}^{i'}, i') - F(u, i')| < \varepsilon/3$ according to the definition of u and the uniform continuity.

i), ii) and iii) imply

$$\begin{aligned} |F(\bar{u}^i, i) - F(u, i)| &\leq |F(\bar{u}^i, i) - F(\bar{u}^{i'}, i')| + |F(\bar{u}^{i'}, i') - F(u, i')| + \\ &+ |F(u, i') - F(u, i)| \leq \varepsilon/3 + \varepsilon/3 + \varepsilon/3 = \varepsilon \quad \text{and (2) is proved.} \end{aligned}$$

The distribution $u \in Q'$, which is a relative distribution, will be assigned to individuals $i \in G_u$. Because of the relation (2) we have distributions which are sufficiently close to the distributions in the hypothetical fair state (X', g') to get an ε -fair state. There only remains to define a feasible social state (X, g) , where the relative utility distributions of the groups $G \in g$ are $u \in Q'$.

We start with $g = \{G_u\}_{u \in Q'}$. For $u \in Q'$ we define a distribution $x_u : G_u \rightarrow R$ according to:

$$x_u(i) = u \left(\frac{v(j : j \leq i, j \in G_n)}{v(G_n)} \right) \quad i \in G_u .$$

A distribution $x : I \rightarrow R$ is defined according to: $x(i) = x_u(i)$ for $i \in G_u$.

Because of $I = \bigcup_{u \in Q'} G_u$ a.e. u is defined a.e. in I .

x has the property $\int_0^1 x(i) di \leq \alpha$ i.e. x is a feasible distribution of the commodity.

We prove that:

$$\begin{aligned} \int_0^1 x(i) di &= \sum_{u \in Q'} \int_{G_u} x(i) di = \sum_{u \in Q'} \int_{G_n} u \left(\frac{v\{j : j \leq i, j \in G_u\}}{v(G_u)} \right) di = \\ &= \sum_{u \in Q'} \int_0^1 v(G_u) \cdot u(i) di = \alpha \sum_{u \in Q'} v(G_u) = \alpha . \end{aligned}$$

A feasible extended social state (X, g) is now defined by $X(i) = (x(i), v)$,

$i \in I$ and $g = \{G_u\}_{u \in Q'}$. $x(i)$ is the quantity of the commodity, and also utility level, assigned to individual i , and v is personal goods which are the same for all individuals.

For $i \in G_u$, $u \in Q'$ it follows from the definition of x and the restriction x_u to G_u that the ethical utility level of i is $F(u, i)$. Relation (2) then shows that (X, g) is an ϵ -fair social state, and the proof is complete. ■

We shall in a later section (14) discuss the assumptions made in $\mathcal{G}_1$ and give an interpretation of the model. However, we conclude this section with two corollaries to illustrate the role played by the continuity assumption of $F : U_I \times I \rightarrow R$ in the variable $i \in I$.

Corollary 1: T1 is true even if we in $\mathcal{G}_1$ change A5 so that $F : U_I \times I \rightarrow R$ is continuous in $U_I \times I_k$ instead of $U_I \times I$, where I_k is measurable, $I = \bigcup_{k=1}^N I_k$ a.e., $I_k \cap I_j = \emptyset$ if $k \neq j$, and F can be extended to a continuous function in $U_I \times \bar{I}_k$.

Proof: There is a finite number N of sets I_k and T1 can be proved if we substitute I for I_k in the same way as earlier. ■

Corollary 2: If we in $\mathcal{G}_1$ accept a countable number of groups $g = \{G_k\}_{k=1}^{\infty}$ in a social state (X, g) , then T1 is true if the continuity assumption in A5 is substituted with:

- 1) $F : U_I \times I \rightarrow R$ is continuous for fixed $i \in I$.
- 2) For fixed $u \in U_I$ $F : U_I \times I \rightarrow R$ is a measurable function which is finite a.e.

Proof: We use Lusin's theorem: If $f : R \rightarrow R$ is a measurable function, which is finite almost everywhere, then for each $\epsilon > 0$ there exists a closed set $A \subset R$ such that the restriction of f to A is continuous and $v(R-A) < \epsilon$.

Define a countable partition of I in the following way, which is possible according to Lusin's theorem:

- i) $I = \bigcup_{k=1}^{\infty} I_k$ a.e., $I_k \cap I_j = \emptyset$ if $k \neq j$, I_k closed.
- ii) $F : U_I \times I_k \rightarrow R$ is continuous for fixed $u \in U_I$.
- iii) $\forall (I - \bigcup_{k=1}^n I_k) < \frac{1}{n}$.

T1 can be proved for each I_k and for a given k we get a finite groupdivision $g_k = \{G_{kj}\}_j$. I is groupdivided according to $g = \{G_{kj}\}_{kj}$ which is countable. ■

14. Fair Division of a Cake Including one Homogeneous Commodity and Personal Goods which may Differ between Individuals

In section 13 we considered a cake to divide which consisted of one commodity which could be transferred between different individuals. We assumed that all individuals were equal or that differences in pure individual properties were not to be taken into account when distributing the single commodity. We shall now relax this assumption and consider a case where differences in personal goods might influence the distribution of the commodity.

We shall consider in principle the same formal model as in sec. 13 and only make small changes. Therefore, we refer to the assumptions and definitions in sec. 13 in most cases.

The main object of this section is to prove a theorem, which is formally exactly the same as T1:13 i.e. the existence of ε -fair social states.

The assumption from sec. 13 about equal personal goods will be relaxed and we shall instead assume that no special connection between personal goods and ethical preferences exists. This will be expressed by an assumption about periodicity in ethical preferences.

The formal model is the following:

Assumption 1: The same as A1:13.

Assumption 2: A position in $\mathcal{P}$ is described by two coordinates (u,v) , where $u \in R$ denotes the quantity of the single commodity to be distributed, and v denotes personal goods.

An individual position is $X(i) = (u(i), v(i)) \in \mathcal{P} \times I$ and a social state is the function $X : I \rightarrow \mathcal{P}$.

Feasible distributions of the single commodity is $S = \{u : u \text{ is a bounded, real integrable function } I \rightarrow \mathbb{R} \text{ with } \int_I u(i) di \leq \alpha, \alpha \in \mathbb{R} \text{ and } |u(i)| \leq L\}$. L is a fixed positive real number.

The set of feasible social states $\mathcal{V}$ is $\{X : X(i) = (u(i), v(i)), u \in S\}$.

Assumption 3: The set I of individuals is divided into N categories of individuals $I_k = (a_k, a_{k+1})$, where $I = \bigcup_{k=1}^N I_k$ a.e., $I_k \cap I_j = \emptyset$ for $k \neq j$ and $v(I_k) > 0$, such that personal goods are equal within a category i.e. $v(i)$ is constant in each I_k .

Subjective preferences $\succsim_i$ for $i \in I$ over $\mathcal{P} \times I$ are described by ordinal utility functions f_i according to $f_i(X(j)) = g_i(u(j), k)$, where k denotes what category I_k that this position belongs to. In the first variable g_i is a continuous and strictly increasing real function from $\mathbb{R}$ onto $\mathbb{R}$.

Assumption 4: The set U_I of distributions of the single commodity is a metric space of integrable real functions $u : I \rightarrow \mathbb{R}$ with the distance function $d(u, u') = \int_0^1 |u(i) - u'(i)| di$ for $u, u' \in U_I$. $U \times I_k$, $k = 1, 2, \dots, N$ are metric spaces with the distance function $d((u, i), (u', i')) = d(u, u') + d(i, i')$ where $d(i, i') = |i - i'|$, $i, i' \in I_k$ for some k .

Assumption 5: Ethical preferences $\{R_i\}_{i \in I}$ over U_I can be represented by an ordinal utility function $F : U_I \times I \rightarrow \mathbb{R}$, which is continuous in $U_I \times I_k$ for each $k = 1, 2, \dots, N$ and can be extended to a continuous function in $U_I \times \bar{I}_k$.

Furthermore F has the following periodicity-property:

$$F(u, i) = F(u, j) \text{ if } i \in I_k, j \in I_\ell \text{ and } \frac{a_{k+1} - i}{a_{k+1} - a_k} = \frac{a_{\ell+1} - j}{a_{\ell+1} - a_\ell}.$$

We denote by $\mathcal{E}_2$ an economy where the assumptions A1, A2, A3 and A4 have been made.

In $\mathfrak{Z}_2$ we define the concepts hypothetical social state, fair hypothetical social state and ε -fair social state just as in sec. 13 (D1, D2, D3 and D4 in sec. 13).

The main theorem of this section is the existence of ε -fair social states. For the proof we will need some lemmas, but we state the theorem now.

Theorem 1: In the economy $\mathfrak{Z}_2$ there exist ε -fair social states for each $\varepsilon > 0$.

When proving T1 in sec. 13 we used increasing distributions as representatives for indifference classes generated by the ethical preferences. The advantage of doing this was that the metric space of uniformly bounded increasing functions was compact and this was proved in lemma 2.

Because of the existence of different categories $\{I_k\}_{k=1}^N$ of individuals, the set of increasing functions can not for certain be chosen as a complete set of representatives for the indifference classes of the ethical preferences, but such a set will instead be the set of those functions which are increasing in each category I_k .

With this change of strategy from the previous section, T1:14 can be proved in a similar way as T1:13.

We denote with K the metric subspace of U_I of functions which are increasing in each I_k , $k = 1, 2, 3, \dots, N$.

Lemma 1: K is a complete set of representatives for the indifference classes of ethical preferences R_i i.e. to each element $u \in U_I$ there is $x \in K$ such that $u \sim_{I_i} x$ for a given $i \in I$.

Proof: According to lemma 1 in sec 13 the restriction $u|_{I_k}$ of $u \in U_I$ to I_k is indifferent to an increasing function $x|_{I_k}$ defined in I_k .

Furthermore, $x(t) = x|_{I_k}(t)$ for $t \in I_k$ is an element in K and according to

T8:5 $x \sim_{I_i} u$. ■

If K^L is the space of bounded distributions in K i.e. $x \in K^L$ if $x \in K$ and $|x(t)| \leq L$ then we can also prove:

Lemma 2: K^L is a compact metric subspace of K .

Proof: K^L can be considered the product of N spaces $K^L = \prod_{k=1}^N K_{I_k}^L$ where $K_{I_k}^L$ is the set of bounded increasing real functions defined in I_k .

According to lemma 2 in sec. 13 $K_{I_k}^L$ is compact. This implies that K^L is also compact. ■

The definition of optimal distributions in S is the same as in sec. 13 (D5) and lemma 3 from this section is also true:

Lemma 3: In $\mathcal{E}_2$ there always exist optimal distributions $\bar{u}^i$ in S for each $i \in I$.

Proof: The proof is in principle the same as the proof of lemma 3 in sec. 12. ■

Proof of theorem 1: The definition of a hypothetical social state $(X'g')$ is exactly the same as in T1:13.

We prove the existence of ε -fair social states by considering the different categories I_k of individuals separately and by considering the space $(K \cap S) \times I$.

For a given I_k , $k = 1, 2, \dots, N$ and $\varepsilon > 0$ we can define a finite set $Q' \subset K \cap S$ of feasible distributions and groups $G_{u, I_k} \subset I_k$ such that:

- i) $I_k = \bigcup_{u \in Q'} G_{u, I_k}$, G_{u, I_k} are disjoint and of measure > 0 .
- ii) $|F(\bar{u}^i, i) - F(u, i)| < \varepsilon$ if $i \in G_{u, I_k}$, $u \in Q'$.

The proof of this is exactly the same as of the corresponding results in T1:13.

Because of the periodicity property of the utility function F in A5 we can

choose the same set Q' for every category of individuals I_k .

If we define $G_u = \bigcup_{k=1}^N G_{u, I_k}$ then:

i) $I = \bigcup_{u \in Q'} G_u$, G_u are disjoint and of measure > 0 .

ii) $|F(\bar{u}^i, i) - F(u, i)| < \epsilon$ if $i \in G_u$, $u \in Q'$.

Finally, an ϵ -fair social state (X, g) is defined as in T1:13.

15. An Interpretation of the Formal Models in sec. 13 and 14 and a Discussion of the Assumptions

The models in the two previous sections describe distribution situations where there is a single commodity to distribute, and where in the first case (sec. 13) all individuals have identical personal goods, and in the second case (sec. 14) personal goods may differ between individuals. We shall in this section discuss the assumptions made in these two models and give an interpretation of the models. We can also compare the results with the general conclusions from sec. 12, where the definition of a fair social state was discussed.

The first model $\mathcal{E}_1$ describes the most simple distribution situation of all; one homogeneous and totally divisible commodity to be distributed fairly among, except for ethical preferences, equal individuals.

Reasonable to assert is that the only fair distribution is the one where every individual has got an equal share of the total amount available of the commodity. However, in our definition of a fair state this is not always the case. A fair state is instead an eventually unequal distribution of the commodity between and a group division of the individuals, which fulfils the conditions of a fair state.

A fair state in $\mathcal{E}_1$ may be interpreted in the following way.

Let g be a groupdivision of the individual set I i.e. $I = \bigcup_{G \in g} G$, $G' \cap G'' = \emptyset$ if $G' \neq G''$ and $G', G'' \in g$. Now assign the same number of identical lottery-tickets L_G to the group $G \in g$ as the number of individuals in G .

The prize-list for the lottery is $\{u(t)\}$, where $u(t)$ is the quantity of the single commodity, and where the variable t takes the same number of values as the number of individuals in G , i.e. there is the same number of prizes as the number of lottery-tickets and the number of individuals in G .

If we define such a lottery for each $G \in g$ then these lotteries generate social states (X, g) , where $X(i) = (u(i), v)$, and $u(i)$ is the prize of individual i and v is the personal goods (which are equal for all individuals in $\mathcal{E}_1$).

A fair social state (X, g) is then characterized by two properties of the lottery:

- 1) if $i \in G$ he has got the lottery ticket L_G and there is no other lottery-ticket $L_{G'}$, $G' \in g$ which he strictly prefers.
- 2) There is no other lottery with groupdivision g' and lottery-tickets $L_{G'}$, $G' \in g'$ such that every individual $i \in G \in g$ and $i \in G' \in g'$ would prefer $L_{G'}$ to L_G and where there is some individual with a strict preference (there is a set of individuals with measure > 0 with strict preferences).

Property 1) of the lottery corresponds to the envy freedom and property 2) expresses the ethical Pareto-optimum property of a fair state.

We notice directly that a state with equal distribution of the commodity to every individual fulfils 1) but not necessarily 2) i.e. there may be a loss of efficiency.

This lottery also provides us with an "economic system" generating fair states.

We have proved (T1:13) the existence of an ε -fair social state. ε is a

measure of how close to an ε -fair state we can find social states, and T1:13 shows that ε -fair states exist for each $\varepsilon > 0$, i.e. we can find social states, which with an arbitrarily good approximation are fair.

In $\mathcal{E}_1$ we have also assumed that there is a continuum of individuals so accordingly the result can be considered an approximation of a finite case with many individuals i.e. if we choose a sufficiently large economy (number of individuals) we can find a social state which is arbitrarily close to a fair social state.

The rest of the assumptions made in $\mathcal{E}_1$ will also need some comment.

A3 only describes a criterion where a strictly greater quantity of the commodity is strictly preferred by all individuals to a smaller quantity.

A4 and A5 express some for all individuals common continuity properties of the ethical preferences.

The ethical preferences are described by an ordinal utility function

$F : U_I \times I \rightarrow \mathbb{R}$ and F is assumed to be continuous in both variables.

Continuity over the space of distributions U_I has been analysed in sec. 6, and we only add that this assumption, exactly like the symmetry property of ethical preferences, is a common restriction for all individual preferences.

The meaning of continuity over I is another. In this case the continuity assumption can be interpreted as if the individual preferences have been assorted, so that individuals with similar ethical preferences are placed in the neighbourhood of each other in the interval $[0,1]$.

An assumption about continuity over I is not essential, which Corollary 2 shows. However, if we assume F to be measurable in I this can also be interpreted as putting individuals with similar ethical preferences close to each other in the interval $[0,1]$.

In A2 we consider as feasible distributions $u(i)$, $i \in I$ only those which are bounded by a real number $L > 0$ i.e. $|u(i)| \leq L$.

This assumption is just like the symmetry and continuity properties of ethical preferences one for all individuals common restriction.

The reason for these remarks about "common restrictions" is, of course, the discussion in sec. 12. There we stated that our theory lies somewhere between the Utilitarian or Rawls's case, where the individuals have to agree on one social utility function, and the case of welfare theory, where only small restrictions were placed on the preferences.

This assumption about uniformly bounded distributions will reflect a case where the marginal utility of the commodity is small or nothing for large quantities. Instead of this assumption about boundedness, we could have assumed that $F(u,i)$ was uniformly bounded e.g.: to each $\epsilon > 0$ there exists a real number L , such that $|F(u,i) - F^L(u,i)| < \epsilon$ where $F^L(u,i) = F(u,i)$ if $|F(u,i)| \leq L$, $F^L(u,i) = L$ if $F(u,i) > L$ and $F(u,i) = -L$ if $F(u,i) < -L$.

Tl:13 could have been proved in this case as well.

Now consider the model $(\mathcal{E}_2)$ in sec. 14. The difference between $\mathcal{E}_1$ and $\mathcal{E}_2$ is the distribution of personal goods among the individuals. In $\mathcal{E}_2$ different individuals may have different personal goods and we have assumed that there is a finite number (N) of different bundles of personal goods.

The distribution of the divisible commodity will depend on the distribution of personal goods.

The model can be interpreted in the following way.

An individual disposes a sum α of money and want to insure himself against different situations. He knows that one of N different events $\{e_1, e_2, \dots, e_N\}$ will happen in the future and the probability that e_k will occur is

$\frac{v(I_k)}{v(I)} = p_k$. If e_k occurs, his personal goods will be those which are characterized by individuals in category I_k in $\mathcal{C}_2$.

Now consider a measurable group division g of $[0,1]$ i.e. $[0,1] = \bigcup_{G \in g} G$, G measurable and $G \cap G' = \emptyset$ if $G \neq G'$ and $G, G' \in g$. Define an insurance L_G for each $G \in g$ such that u_G is a distribution of money in G with $\int_G u_G(t) dt = \alpha \cdot v(G)$.

u is the total distribution in $[0,1]$, i.e. $u(t) = u_G(t)$ if $t \in G$.

We furthermore assume that $v(G \cap I_k) > 0$ for each I_k and each $G \in g$.

We use the same group division g on the individual set $I = [0,1]$ and assign the insurance L_G to an individual i if $i \in G$.

This insurance system $\{L_G\}_{G \in g}$ will generate social states (X, g) with $X(i) = (u(i), v(i))$ and where $u(i)$ is the outcome of the insurance L_G for $i \in G$ and $v(i)$ the personal goods of this individual.

Fair social states are generated if we can find an insurance system

$\{L_G\}_{G \in g}$ where:

1) if $i \in G$ he has got insurance L_G and there is no other insurance $L_{G'}$, $G' \in g$ which he strictly prefers.

2) There is no other insurance system $\{L_{G'}\}_{G' \in g'}$ such that for all individuals $L_{G'}$ is at least as good as L_G where $i \in G \in g$ and $i \in G' \in g'$ and where this relation is true with strict preference for a set of individuals with measure > 0 .

Property 1) is of course the envy freedom of a fair state and 2) expresses the ethical Pareto-optimum property.

We also notice that a special insurance L_G assigns to each event e_k not a special sum of money, but a distribution $u_{G \cap I_k}$. The normal case of insurances is perhaps to assign the mean $\int_{G \cap I_k} u_{G \cap I_k}$ of the distribution to the event e_k .

If we do so we can achieve our fair state by using the machinery from $\mathcal{E}_1$ afterwards.

In any case we can see $A_k = \frac{1}{v(G \cap I_k)} \cdot \int_{G \cap I_k} u_{G \cap I_k}$ $k = 1, 2, \dots, N$ as a fair distribution of resources in regard to the N different kinds of personal goods for the group G . A_k is what individuals in G get on an average if event e_k should occur.

This insurance system generates social states (X, g) and T1:14 shows the existence of an at least approximative fair social state.

To prove this we have made some assumptions, which in most cases are the same as in $\mathcal{E}_1$, but there is one difference which needs a comment.

A5 includes an assumption of periodicity in the ordinal representation F of individuals' ethical preferences. This may be interpreted as a condition that no special connection exists between an individual's ethical preferences and the future category to which he will belong. Such an assumption is natural if the events e_k , $k = 1, 2, \dots$ occur randomly among the individuals.

Finally, we can compare the results from sec. 13 and sec. 14 to the general discussion in sec. 12 about the existence of mechanisms in an economy to reveal ethical preferences.

The general conclusion was that there were varying possibilities of revealing preferences; it was difficult to take into consideration the preferences of a special individual to decide his fair welfare level if something should occur while he is still a child. For events which occur "later" in the individual's life it is possible to define an insurance system which reveals his ethical preferences.

This later type of distribution mechanisms is the one described by the models $\mathcal{E}_1$ and $\mathcal{E}_2$. Especially the assumption of periodicity of the preferences in $\mathcal{E}_2$ underlines this.

Chapter IV. FAIR DISTRIBUTIONS OF INDIVISIBLES

16. Introduction

In chapter III we have studied the problem of fair division of a social cake. To get a theory of what is meant by a fair welfare distribution in a society this cake will necessarily have to include not only goods, which can be transferred from one individual to another, but also personal goods, i.e. individual properties like different kinds of abilities, health etc. We also pointed out that an economic system generating fair social states have to hold certain distribution mechanisms or subsystems, whose purpose is to produce distributions which take the ethical preferences of the different individuals into account.

An insurance system is such an example. A condition for these subsystems to generate fair social states is intuitively that all individuals are treated equally by the systems or have the same possibilities of using the systems. This can often be equivalent to an equal-division problem. Take for instance a simple economy where there is a cake of commodities to divide, and where some of these commodities are such subsystems, e.g. insurances of different kinds and lotteries. Then an equal division, in some sense, of this cake may be a step on the way to get a fair social state, according to the definition in the previous chapter. It is, of course, the envy-free property of a fair social state that we try to secure. Thus, this leads us to study the problem of fair division of cakes which do not include personal goods, and where all individuals are equally qualified to receive any share.

The main difference between division problems of this kind and the division of a social cake, considered in the previous chapters, is that subjective preferences instead of ethical preferences can now be used. We have earlier pointed out that there may be a loss of efficiency by using subjective preferences instead of ethical ones. This loss, however, is not grave, because

the shares received are used as an input in the economic system to determine the "final" consumption. We are therefore primarily interested in satisfying the envy-free property of a fair social state rather than the ethical Pareto-optimum property.

The problem of fair division of a fixed amount of certain goods among a given number of individuals has been studied in Varian (1974). An essential restriction in his model, however, is an assumption about totally divisible commodities. The existence of indivisibles is in many cases significant for an economy. We shall in the next chapter consider in detail a model consisting of a certain amount of money, which is totally divisible, and a set of jobs, which are indivisible, and define fair division of such a cake. The distribution of money over jobs will be defined as a fair wage-structure. It is also possible to consider as indivisibles houses, flats, cars etc. All these important distribution problems are not in general possible to examine in models with only divisible goods.

In this chapter we shall consider division problems where the existence of indivisibles is essential. Indivisibles can be of many different kinds, but we shall restrict ourselves to a case where each individual is assumed to consume exactly one of these goods and in addition to that a certain amount of a divisible good. We shall also consider a case where the indivisibles have some divisible properties or characteristics. We thereby have a case which is formally very much like Varian's case. In Varian a commodity bundle is considered as a point in a topological vector space and our corresponding objects will be the indivisibles which are considered as a point in a space of characteristics. This space is a topological space but not a vector space. The difference may seem small but will cause important consequences for the price-systems which are often associated with Pareto-optimal allocations. This is, of course, of interest in general and not only in a theory of fairness.

A theory of fair division is always built up around a criterion which de-

defines "equal division". We have in sec. 8 discussed possible forms of such criteria, and in the definition of a fair state in sec. 10 we used the fairness criterion. The division situation in this chapter is formally very much like the situation in sec. 10, we shall only substitute ethical preferences for subjective preferences. Consequently it is still natural to consider as a necessity for equal division, that distributions fulfil the fairness criterion. To each division of the commodity-cake and to each individual we associate a division of a utility-cake, which has to be divided fairly according to the fairness criterion.

The sections in this chapter will have the following contents. In sec. 17 we define the formal model which will be used in different versions in this chapter. We also compare with a case of fair division of divisible goods, Varian's model. A main theorem in this chapter about the existence of fair allocations is formulated in sec. 18 and proved in sec. 20. Preferences used are described and characterized in different ways in sec. 19. In sec. 21 we consider some alternative models to the model from sec. 18, and in sec. 21 a case with incomplete preferences are being considered. Sec. 23 contains a case where the indivisibles have certain divisible properties, and an existence theorem about fair allocations is proved. We also define fair equilibrium allocations which is a stronger concept than fair allocations. Finally, we shall compare our model from sec. 18, where a case with a finite number of indivisibles and one divisible good was considered, with a similar model, which has been analysed in Gale (1960). The main difference is the assumptions made about preferences. In Gale's model preferences are in a certain sense linear. This restriction will not be used in the models in this chapter.

17. A General Model

In this section we define a general model which will be used in this chapter.

but which will later be complemented with special assumptions for special situations. We also compare with Varian's model and results.

An economy $\mathcal{E}$ is defined by the following assumptions and concepts:

Assumption 1: There is a finite set $I = \{i_1, i_2, \dots, i_N\}$ of N individuals.

Assumption 2: There is a finite set $A = \{a_1, a_2, \dots, a_N\}$ of N indivisibles.

Each element $a \in A$ can be represented by a point in a space of characteristics C , which is a topological space.

We shall also use the concept allocation and mean a bijection $x : I \rightarrow A$ i.e. each individual is associated with an element in A . We often mean by $x(i)$ the representation in C of the element in A assigned to $i \in I$ i.e. $x(i)$ is a point in $C \times I$.

The set of feasible allocations will be denoted F .

Assumption 3: Each individual $i \in I$ has preferences $\succsim_i$ over C . The relation $\succsim_i$ is supposed to be a pre-ordering i.e. a reflexive and transitive binary relation in C . $\succ_i$ and $\sim_i$ are the corresponding strict and indifference relations i.e. $a \succ_i b$ if $a \succsim_i b, b \not\succsim_i a$ and $a \sim_i b$ if $a \succsim_i b, b \succsim_i a$.

In $\mathcal{E}$ we shall study the existence of fair allocations under different complementary assumptions.

Definition 1: An allocation $x \in F$ is Pareto-optimal if no allocation $y \in F$ exists such that:

- 1) $y(i) \succsim_i x(i) \quad \forall i \in I$
- 2) $\exists i \in I$ such that $y(i) \succ_i x(i)$.

Definition 2: An allocation x is envy-free if for all $i, j \in I$ either $x(i) \succsim_i x(j)$ or $x(i)$ and $x(j)$ is not comparable under the relation $\succsim_i$.

Definition 3: An allocation $x \in F$ is fair if x is Pareto-optimal and envy-free.

Before specifying the model further we can compare $\mathcal{E}$ with the corresponding model in Varian (1974). That model will be denoted $\mathcal{E}_1$. Besides the assumptions made in $\mathcal{E}$ we have to assume the following to get $\mathcal{E}_1$.

- 1) C is assumed to be $R_+^n = \{x : x \in R^n \text{ and } x \geq 0\}$ which is the commodity space. R_+^n is a subset of the topological vector space R^n with the usual topology.
- 2) Feasible allocations are those, where $\sum_i x(i) \leq w$ and $w \in R_+^n$ is the total amount of commodities available.
- 3) Preferences are complete, monotone and continuous pre-orderings.

In $\mathcal{E}_1$ fair allocations exist under certain supplementary assumptions and we state this as two theorems without proofs. The proofs can be found in Varian (1974).

Theorem 1: If preferences $\succsim_i$ are convex then fair allocations exist in $\mathcal{E}_1$.

Theorem 2: If in $\mathcal{E}_1$ no two Pareto-optimal allocations x and y exist such that $x(i) \sim y(i) \quad \forall i \in I$ then at least one fair allocation exists.

If we want to consider large economies, where there is a continuum of individuals, then the existence of fair allocations can be proved even if the assumptions about preferences in T1 and T2 are relaxed. This is also proved in Varian (1974).

18. The Definition of $\mathcal{E}_2$.

Now consider the economy $\mathcal{E}$ and define $\mathcal{E}_2$ by adding the following assumptions (A1, A2, A3 and A4).

Assumption 1: Each element $a \in A$ is characterized by two coordinates (e, p) where e is an element in $E = \{e_1, e_2, \dots, e_N\}$ and p is a real number (we do not exclude that $e_k = e_j$ for some k, j). An indivisible $a_k \in A$ corresponds to the element (e_k, p) with $e_k \in E$ and $C = E \times R$.

A1 shows that if we interpret an element $a \in A$ as a commodity then e_k is "the indivisible element" and p is a divisible part, which is hanged on a . If a is a car then p can e.g. be the quantity of petrol in the tank. Thus, it is possible to change one of the characteristics defining a .

Assumption 2: C is a metric space with the metric $d((e', p'), (e'', p'')) = d'(e', e'') + d''(p', p'')$ where $d'(e', e'') = 0$ if $e' = e''$ and equal 1 if $e' \neq e''$ and $d''(p', p'') = |p' - p''|$.

C is also partially ordered by a relation $\geq$ according to $(e, p') \geq (e, p'')$ if and only if $p' \geq p''$.

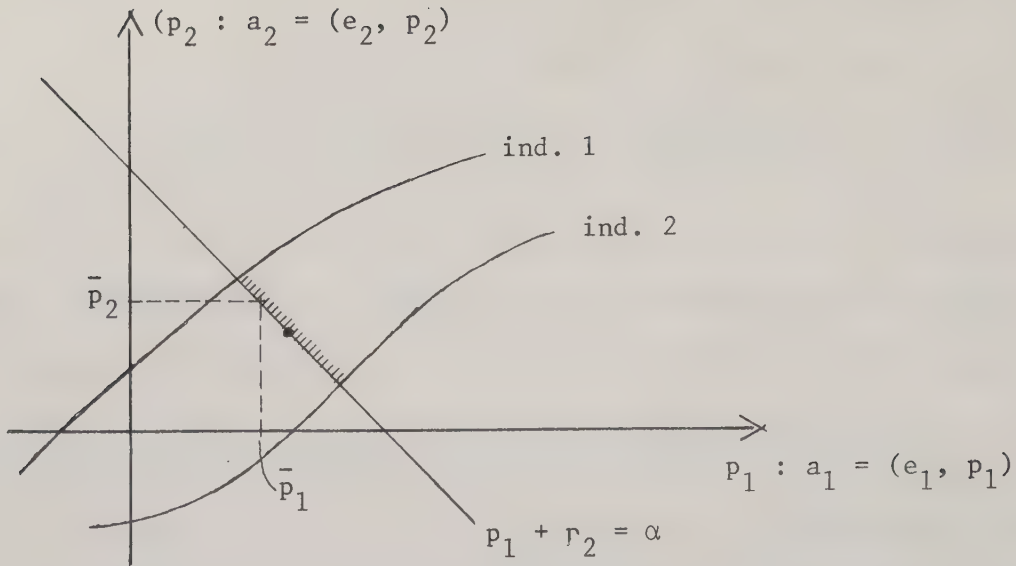
d is obviously a metric on C and $\geq$ is a partial order relation in C .

Assumption 3: The preference relation $\succsim_i$ of individual $i \in I$ is a continuous, strictly monotone and complete pre-ordering over C .

The definitions of the continuity and strictly monotone properties are the usual ones. A preference relation $\succsim$ is continuous if the sets $\{a : a \succsim a', a \in C\}$ and $\{a : a \succ a', a \in C\}$ are closed and open respectively for all $a' \in C$. A preference relation $\succsim$ is strictly monotone if for $a, a' \in C$ $a \geq a'$ implies that $a \succsim a'$ and $a > a'$ implies that $a \succ a'$.

Assumption 4: Feasible allocations are $F = \{x : x(i) = (e(i), p(i)) \text{ where } e \text{ is a bijection from } I \text{ to } E \text{ and } p \text{ a real function } I \rightarrow R \text{ with } \sum_i p(i) \leq \alpha \text{ for a given real number } \alpha\}$.

F can be seen as a set of distributions partly of the set of indivisibles E and partly of a totally divisible commodity, which is available in the total amount α . However, no special assumptions about the sign of $p(i)$ are made. The distribution situation with these feasible allocations can be illustrated graphically in the case of two elements in $E = \{e_1, e_2\}$.



The quantity of the divisible commodity p_k associated with element $e_k \in E$ has been marked on the axis and feasible combinations of p_1 and p_2 are those on the line $p_1 + p_2 = \alpha$.

The preferences of the individuals divide the plan into two areas. One such area represents combinations (p_1, p_2) where a_1 is at least as good as a_2 , and in the other area a_2 is at least as good as a_1 . The two lines mark those (p_1, p_2) where i_1 and i_2 respectively are indifferent between a_1 and a_2 . We shall later prove this statement exactly. In the figure we have also marked a point $(\bar{p}_1, \bar{p}_2)$ where $a_2 \succ_{i_1} a_1$ and $a_1 \succ_{i_2} a_2$. This means that an allocation $x(i_1) = a_2$ and $x(i_2) = a_1$ is envy-free.² Because of the strictly monotone property of the preferences x is also Pareto-optimal and thus a fair allocation.

The existence of fair allocations seems to be easy to prove in this case with only two indivisibles. However, if the number of indivisibles are greater than two, the proof is rather extensive, and the next sections will be used for this purpose. Primarily we prove the existence for this type of feasibilities, but it is also easy to extend the proof to cases where the

feasible set has been continuously deformed. This problem, and what will happen if preferences are not complete, will be considered in later sections. We conclude this section by stating formally the existence theorem.

Theorem 1: In $\mathcal{E}_2$ there exists fair allocations for each integer N , the number of indivisibles and the number of individuals, and for each $\alpha \in \mathbb{R}$, the amount of the divisible commodity available, if for some $\alpha \in \mathbb{R}$ and each $i \in I$ there exist an allocation $x^i = (e^i, p^i)$ with $\sum_{j=1}^N p^i(j) = \alpha$ such that $x^i(j) \sim_i x^i(k) \quad \forall j, k \in I$.

19. Characterization of Preferences

In this section we shall derive some alternative descriptions of the preferences in $\mathcal{E}_2$. Therefore we consider preferences $(\succsim)$ in general and assume $\succsim$ to be a continuous, strictly monotone and complete pre-ordering over the metric space $C = E \times \mathbb{R}$.

Every continuous and complete pre-ordering on a separable topological space X can be represented by an ordinal continuous utility function u i.e.

$u: X \rightarrow \mathbb{R}$, $u(x) \geq u(y) \Leftrightarrow x \succsim y$ for $x, y \in X$ and u continuous in X . For a proof of this we refer e.g. to Debreu (1964). With this in mind it is easy to prove:

Theorem 1: Preferences $\succsim$ in $\mathcal{E}_2$ over C can be represented by an ordinal continuous and monotone utility function $u: C \rightarrow \mathbb{R}$.

Proof: C is a metric space which is the product of two separable metric spaces E and $\mathbb{R}$. Consequently C is also separable, and a continuous utility function u exists. The monotone property of the preferences $\succsim$ is directly transformed to u . ■

The utility function u can more explicitly be written as $u = u(e, p)$ where $e \in E$ and $p \in \mathbb{R}$. For fixed $e \in E$ u is continuous in the variable p .

Because $C = E \times R$ is not a connected space unless E contains only one element, the range of u need not be an interval. However, C is the union of N components, maximal connected subspaces of C , i.e. $C = \bigcup_{e \in E} \{e\} \times R$. The restriction of u to $\{e\} \times R$ for $e \in E$ is a continuous real function defined on a connected space and the range is consequently an interval. We have now proved the following theorem.

Theorem 2: The range V_u of u is the union of N (open) intervals : $V_u = \bigcup_{k=1}^n I_k$, where I_k is the range of the restriction of u to $\{e_k\} \cup R$, $e_k \in E$.

Preferences $\succsim$ over $C = E \times R$ and allocations in C can be described by sets and points in R^N . Allocations have been defined as functions $x : I \rightarrow C$ and we define a function T from the set of allocations to R^N according to $T(x) = p$ if $x(i) = (e(i), p(i))$ i.e. p_k is the real number assigned to $e_k \in E$. $T(x)$ is the distribution of the divisible good represented by a point in R^N .

Preferences $\succsim$ over $C = E \times R$ induce preferences over the different elements p_k in elements $p \in R^N$. When no misunderstandings are possible we write $p_k \succsim p_j$ when $(e_k, p_k) \succsim (e_j, p_j)$. It is, of course, essential that by p_k is meant not only the real number represented by p_k , but also that this number is assigned to element $e_k \in E$. A partial description of preferences over $C = E \times R$ can be received by stating "preferred sets" in R^N according to:

$$M_k = \{p : p_k \succsim p_j \ \forall j, \ p \in R^N\}$$

M_k is the set of the distributions of the divisible good where (e_k, p_k) is at least as good as (e_j, p_j) for all j . Obviously $R^N = \bigcup_{k=1}^N M_k$ but M_k may be the empty set for some k . In the same way we introduce "strictly preferred sets":

$$m_k = \{p : p_k \succ p_j \ \forall j, \ p \in R^N\}.$$

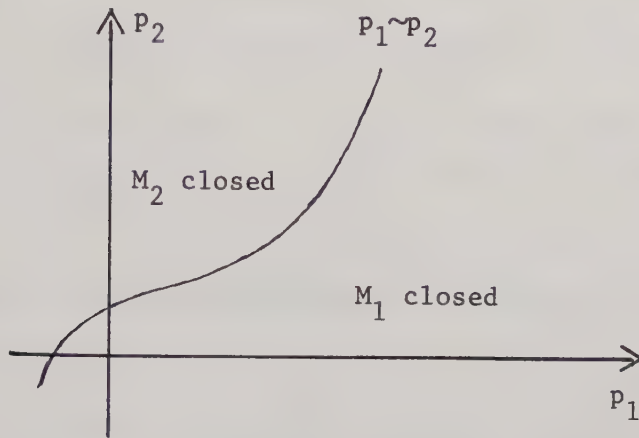
Theorem 3: The sets M_k and m_k are closed and open subsets of R^N respectively.

Proof: Suppose $p^r \in M_k$ and $p^r \rightarrow \bar{p} \in R^N$ when $r \rightarrow \infty$. We have to prove that $\bar{p} \in M_k$.

$$p^r \in M_k \Leftrightarrow u(e_k, p_k^r) \geq u(e_j, p_j^r) \quad \forall r, j.$$

The continuity of u implies that this relation is true if we substitute p^r with $\bar{p}$ i.e. $\bar{p} \in M_k$ and thus M_k is closed. If $M_k = \emptyset$ M_k is also closed.

m_k is open because m_k is the complement to $\bigcup_{j \neq k} M_j$ which is closed; the union of a finite number of closed sets. ■



Theorem 4: $M_k \neq \emptyset \Rightarrow$ For each hyperplane in R^N $\pi : \bar{p} \cdot q = \alpha$, where $q > 0$ is a fixed element in R^N with all coordinates strictly greater than zero $\alpha \in R$ and $p \in R^N$, we always have $\pi \cap M_k \neq \emptyset$.

Proof: $M_k \neq \emptyset \Rightarrow \exists \bar{p} \in M_k$ such that $\bar{p}_k \gtrsim \bar{p}_j \quad \forall j$. If $\bar{p} \cdot q = \alpha$ then $\bar{p} \in \pi$.

Otherwise: 1) $p \cdot q > \alpha$ or 2) $p \cdot q < \alpha$.

1) Define $\bar{p}_j = \bar{p}_j \quad \forall j \neq r$ where $r \neq k$ and $\bar{p}_r = \bar{p}_r - \frac{\bar{p} \cdot q - \alpha}{q_r}$. Thus $\bar{p} \cdot q = \alpha$ and, because of the monotonicity of the preferences $\bar{p}_k \gtrsim \bar{p}_j \quad \forall j$ i.e. $\bar{p} \in M_k \cap \pi$ which consequently is not empty.

2) We only have to increase $\bar{p}_k$ to $\bar{p}_k$ and put $\bar{p}_j = \bar{p}_j$ for $j \neq k$ so that $\bar{p} \in \pi$. ■

Theorem 5: $M_k \neq \emptyset \quad \forall k \Rightarrow$ in each hyperplane $\pi : p \cdot q = \alpha$, where $q > 0$ is fixed and $p \in R^N$, exists exactly one point $\bar{p} \in \pi$ such that $\bar{p}_j \sim \bar{p}_k \quad \forall j, k$.

Proof: Consider a hyperplane $\pi : p \cdot q = \alpha$, q fixed and > 0 , $p \in R^N$.

According to T4, $M_1 \cap \pi \neq \emptyset$. Put $\bar{p}_1 = \inf\{p_1 : p \in M_1 \cap \pi\}$. In that case $-\infty < \bar{p}_1 < \infty$ because otherwise there is an infinite sequence $p^k \in M_1 \cap \pi$ each that $p_1^k \rightarrow -\infty$ when $k \rightarrow \infty$. This also implies that there is a subsequence $\{k_r\}$ of $\{k\}$ such that $p_j^{k_r} \rightarrow +\infty$ when $r \rightarrow \infty$ i.e. $p_1 \succ p_j$ for all values of p_1 and p_j . However, this is a contradiction to $M_j \cap \pi \neq \emptyset$ (T4).

Now put $\bar{p}_k = \sup\{p_k : \bar{p}_1 \approx p_k\}$. Just as in the above, $\bar{p}_k < \infty$. From the continuity of the preferences also follows that $\bar{p}_1 \sim \bar{p}_k \quad \forall k$.

There only remains to prove that $\bar{p} \in \pi$. Assume the reverse, i.e. that

1) $\bar{p}q > \alpha$ or 2) $\bar{p}q < \alpha$.

1) We put $\bar{\bar{p}}_1 = \bar{p}_1$ and $\bar{\bar{p}}_k < \bar{p}_k$ for $k > 1$ so that $\bar{\bar{p}} \in \pi$.

In that case $\bar{\bar{p}}_1 \succ \bar{\bar{p}}_k$ for $k > 1$ i.e. $\bar{\bar{p}} \in m_1$ (the strictly preferred set) and according to T3 m_1 is open. This is, however, a contradiction to the construction of $\bar{p}_1 = \bar{\bar{p}}_1$ as an infimum.

2) We put $\bar{\bar{p}}_k = \bar{p}_k$ for $k > 1$ and $\bar{\bar{p}}_1 > \bar{p}_1$ so that $\bar{\bar{p}} \in \pi$. In that case

$\bar{\bar{p}}_1 \succ \bar{\bar{p}}_k \quad \forall k > 1$ i.e. $\bar{\bar{p}} \in m_1$ which is open. This is a contradiction to the construction of $\bar{x}_k$ as a supremum.

We have now proved the existence of $\bar{p}$, but $\bar{p}$ is obviously also unique and the proof is complete. ■

T5 tells us that for all hyperplanes π with strictly positive normal there is always a point $\bar{p} \in \pi$ with total indifference between the indivisibles

$$(e_k, \bar{p}_k) \sim (e_j, \bar{p}_j) \quad \forall k, j.$$

An equivalent formulation is that $(\bigcap_{k=1}^N M_k) \cap \pi \neq \emptyset$ for all such hyperplanes π .

Now denote with $\mathcal{J}$ the set of all indifferences in R^N i.e.

$$\mathcal{J} = \{p : p_k \sim p_j \quad \forall k, j \text{ and } p \in R^N\}.$$

To each element $p \in \mathcal{J}$ we can associate exactly one utility level

$t = u(e_k, p_k)$ where p_k is one of the coordinates in p . Thus we have a functional relation $t = t(p)$, $p \in \mathcal{J}$. The strict monotone property of the pref-

erences implies that $t(p)$ is a strictly increasing function and thereby injective.

We have earlier defined the range of the restriction of u to the set

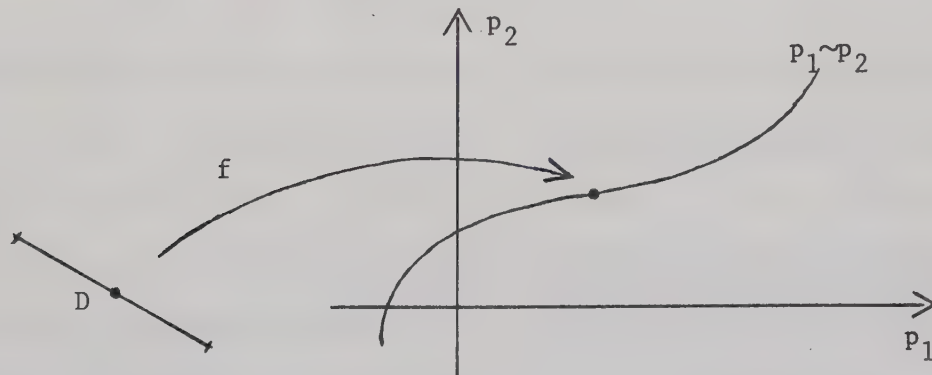
$\{e_k\} \times \mathbb{R}$ to be the open interval I_k . Obviously the range of $t(p)$ will be the interval $D = \bigcap_{k=1}^N I_k$. This means that the inverse $f = t^{-1}$ of $t(p)$ can be defined as a function $f : D \rightarrow \mathbb{R}^N$.

f assigns to each utility level in D the indifference distribution p in $\mathbb{R}^N$.

We call f the indifference function. (f will, of course, depend on the choice of the ordinal utility function u to represent preferences $\succsim$, but the range of f is uniquely determined).

Theorem 6: The indifference function $f : D \rightarrow \mathbb{R}^N$ is continuous and strictly increasing.

Proof: This is an immediate consequence of T5 (the uniqueness), the strictly monotone property, and the continuity property of the preferences. ■



The indifference function f is unbounded in at least one coordinate (T5) if $D \neq \emptyset$. If f should be unbounded in every coordinate it would describe the preferences $\succsim$ completely according to:

$$p_k \succsim p_j \Leftrightarrow \{f_k(t) = p_k \Rightarrow f_j(t) \geq p_j\}.$$

This property can be achieved by extending the indifference function. Denote by $\mathcal{D} = \bigcup_{k=1}^N I_k$ and $\bar{\mathbb{R}} = [-\infty, \infty]$.

Theorem 7: There exists a generalized indifference function $f : \mathcal{D} \rightarrow \bar{\mathbb{R}}^N$ such that the restriction of f to D is the indifference function and such that preferences $\succsim$ are defined completely according to:

$$p_k \succsim p_j \Leftrightarrow \{f_k(t) = p_k \Rightarrow f_j(t) \geq p_j\}$$

Furthermore the components f_k , $k = 1, 2, \dots, N$ of f are continuous and strictly increasing when $f_k(t) \neq \pm \infty$.

Proof: If $f = (f_1, f_2, \dots, f_N)$ the different components f_k are defined according to:

$$f_k(t) = p_k \text{ such that } u(e_k, p_k) = t \text{ for } t \in I_k.$$

$$f_k(t) = \infty \text{ if } t \in \mathcal{D} \text{ and } t \geq \sup t', t' \in I_k.$$

$$f_k(t) = -\infty \text{ if } t \in \mathcal{D} \text{ and } t \leq \inf t', t' \in I_k.$$

It is easy to check that f defined in this way fulfils the requirements in the theorem. ■

We conclude this section by observing that a certain kind of incompleteness in the preferences can be allowed when this generalized indifference function is used. There will be no problem if preferences are defined and complete over sets $E' \times \mathbb{R}$ where $E' \subset E$. If some of the indivisibles are not ranked, the corresponding components in f will disappear.

20. A Proof for the Existence of Fair Allocations in $\mathcal{E}_2$.

We shall in this section give a proof for the existence of fair allocations in $\mathcal{E}_2$ i.e. a proof for T1 in sec. 18. For the proof we will need a number of lemmas (L1, ..., L5), which we first state and prove.

First we will give a simple formulation of Pareto-optimality in a fair allocation.

Lemma 1: If x is an envy-free allocation in $\mathcal{E}_2$ and $\sum_{k=1}^n p_k = \alpha$, $T(x) = p$ then x is fair.

Proof: We have to prove that x is Pareto-optimal. Suppose the contrary i.e. the existence of a feasible allocation y such that $y(i) \succeq_i x(i) \quad \forall i$ and $y(i) \succ_i x(i)$ for some i .

Suppose that the distribution of the divisible commodity in R^N is p and p' for x and y i.e. $T(x) = p$ and $T(y) = p'$.

Now consider an arbitrary element $e_k \in E$ and the corresponding p_k and p'_k .

There is $i \in I$ such that $y(i) = (e_k, p'_k)$. In that case we have:

$$(e_k, p'_k) = y(i) \succeq_i x(i) \succeq_i (e_k, p_k) \Rightarrow p'_k \geq p_k.$$

This follows from the assumption of y , and from that x is envy-free, and from the strict monotone property of the preferences. For different k we get different i and for some k we get the individual for which $y(i) \succ_i x(i)$. In that case $p'_k > p_k$.

Thus: $\alpha = \sum_{k=1}^W p_k < \sum_{k=1}^N p'_k$ and y is not feasible.

This is a contradiction and the lemma is proved. ■

Tl:18 will be proved by induction over the number N of individuals and indivisibles in $\mathcal{C}_2$. Therefore in the rest of this section we assume that Tl:18 is true for each integer $n \leq N$ and prove that this assumption implies that Tl:18 is also true for $N+1$. According to L1 every envy-free allocation is fair for some α . Therefore we need not in general make any difference between a fair and an envy-free allocation.

A special notation and some definitions are needed for this section. The function T maps an allocation x on an element p in R^n , the distribution of the divisible commodity. We shall also consider subsets $m \subset E$ and to each such set associate a space $L_m = \{p_m : p_m = (p_{k_1}, p_{k_2}, \dots, p_{k_n})\}$ where k_j is increasing in j and $m = \{e_{k_1}, e_{k_2}, \dots, e_{k_n}\}$. L_m is considered to be a metric space with the usual topology of R^n . For an allocation x and a group $G \subset I$ x_G is the restriction of x to G . If $m = \{e_k : e_k \in E \text{ and } x_G(i) = (e_k, p_k) \text{ for}$

some $i \in G$ we extend T to be defined for x_G according to $T(x_G) = p_m$ where p_m is the element in L_m which represents the distribution of the divisible commodity in x_G .

Preferences $\succsim_i$ over $E \times R$ induce preferences $\succsim_i$ (the same notation) over $L = \bigcup_{m \in A} L_m$, the set of all possible distributions of the divisible commodity, according to : $p'_m, p''_m \in L$ and $p'_m \succsim_i p''_m \Leftrightarrow \exists k$ such that $e_k \in m'$ and $(e_k, p'_k) \succsim_i (e_j, p''_j) \forall e_j \in m''$ (or shorter $p'_k \succsim_i p''_j \forall e_j \in m''$).

To precise the economy $\mathcal{E}_2$ further we can use the notation $\mathcal{E}_2(\alpha, n)$, where α is the amount of the divisible commodity available and n is the number of individuals and indivisibles.

Lemma 2: 1) x is a fair allocation in $\mathcal{E}_2(\alpha, n)$ and $T(x) = p$.

2) $m \subset A$, $m \neq \emptyset$ and $m^* = A - m \neq \emptyset \Rightarrow p_m \sim_i p_{m^*}$ for some $i \in G = \{i : i \in I \text{ and } x(i) = (e_k, p_k) \text{ for } e_k \in m\}$.

1), 2) $\Rightarrow$ If x' is a fair allocation in $\mathcal{E}_2(\alpha', n)$ for some α' and $T(x') \neq T(x)$ then $T(x') \geq T(x)$ or $T(x') \leq T(x)$.

Proof: Suppose that L2 is not true i.e. that there is a fair allocation x' in $\mathcal{E}_2(\alpha', n)$ for some α' such that:

$$p'_k > p_k \text{ for } k : e_k \in m \neq \emptyset . \text{ (or } p'_k < p_k)$$

$$p'_k \leq p_k \text{ for } k : e_k \in m^* \neq \emptyset . \text{ (or } p'_k \geq p_k)$$

and where $T(x) = p$, $T(x') = p'$.

We put $G = \{i : x(i) = (e_k, p_k) \text{ for } e_k \in m\}$.

Because of the strict monotone property of the preferences and because x and x' are envy-free $p'_m \succ_i p_m \succsim_i p_{m^*} \succsim_i p'_{m^*} \forall i \in G$.

According to condition 2) in this lemma there is some individual $i \in G^*$ so that $p_m \sim_i p_{m^*}$. For this individual the relation $p'_m \succ_i p'_{m^*}$ must also be true. Thus, there are more individuals for which $p'_m \succ_i p'_{m^*}$ than there are elements

in m , and x' is therefore not envy-free, which is a contradiction.

The case when $p'_k < p_k$ for $e_k \in m$ and $p'_k \geq p_k$ for $e_k \in m^*$ is treated in the same way. The lemma is now completely proved. ■

L2 tells us that the conditions 1) and 2) are sufficient to get a unique distribution p of the divisible commodity, for a given total amount α , in a fair allocation. However, there may be several allocations x with $T(x) = p$. Furthermore, if p' is the distribution corresponding to another fair allocation x' ($T(x') = p'$) then p' must be situated in the cone with p as a vertex.

Lemma 3: Under the assumptions in T1:18 there are, to each fair allocation x and to each neighbourhood S to $T(x)$, fair allocations x' and x'' with $T(x'), T(x) \in S$ and $T(x') \leq T(x) \leq T(x'')$, $T(x') \neq T(x) \neq T(x'')$.

Proof: Put $p = T(x)$, $p' = T(x')$ and $p'' = T(x'')$.

For an arbitrary $m \subset A$, $m \neq A$ and $m \neq \emptyset$ we have one of the following cases:

- 1) $p_m \sim_i p_{m^*}$ for some $i \in G = \{i : i \in I \text{ and } x(i) = (e_k, p_k) \text{ with } e_k \in m\}$.
- 2) $p_m \succ_i p_{m^*} \quad \forall i \in G$.

If 1) is true for every $m \subset A$ with $m \neq A$ and $m \neq \emptyset$ then L3 follows from L2 and T1:18.

I. $T(x') \leq T(x)$.

If there is m such that 2) is true we can choose such an m so that m has a minimal number of elements. (m has at least one element!)

Now consider only elements in m and individuals $i \in G = \{i : i \in I \text{ and } x(i) = (e_k, p_k) \text{ with } e_k \in m\}$. The conditions in L2 is fulfilled because of the choice of a minimal number of elements in m .

According to T1:18 and L2 there is, to every neighbourhood S' to $p_m = T(x_G)$, a fair allocation x'_G with $T(x'_G) = p'_m \in S'$ and $p'_m \leq p_m$, $p'_m \neq p_m$.

If S' is chosen sufficiently small, the continuity of the preferences implies that $p'_m \succ_i p_{m*} \quad \forall i \in G$ because $p_m \succ_i p_{m*} \quad \forall i \in G$ according to the definition of G and m .

We can now construct an allocation x' according to:

$$x'(i) = x'_G(i) \text{ for } i \in G.$$

$$x'(i) = x(i) \text{ for } i \in G^*.$$

If S' has been chosen sufficiently small then $T(x') \in S$ for a given S .

Furthermore, x' is a fair (envy-free) allocation with $T(x') \leq T(x)$, $T(x') \neq T(x)$. The last condition follows from the construction of x' and the envy-free property is guaranteed by the relations: $p'_m \succ_i p_{m*} = p'_{m*}$ for $i \in G$ and $p'_{m*} = p_{m*} \succsim_i p_m \geq p'_m$ for $i \in G^*$.

$$\text{II. } T(x) \leq T(x').$$

This case is similar to case I. If there is m such that 2) is true we can choose such an m so that m has a maximal number of elements. ($m \neq A!$)

Now consider only elements in $m^* = E - m$ and the group $G^* = \{i : i \in I \text{ and } x(i) = (e_k, p_k) \text{ with } e_k \in m^*\}$. The conditions in L2 are fulfilled because of the choice of a maximal number of elements in m .

According to T1:18 and L2 there is, to every neighbourhood S'' to $p_{m*} = T(x_{G*})$, a fair allocation x''_{G*} with $T(x''_{G*}) = p''_{m*} \in S''$ and $p_{m*} \leq p''_{m*}$, $p_{m*} \neq p''_{m*}$.

If S' is chosen sufficiently small the continuity of the preferences implies that $p_m \sim_i p''_{m*} \quad \forall i \in G = I - G^*$.

We can now construct an allocation x'' according to:

$$x''(i) = x''_{G*}(i) \text{ for } i \in G^*.$$

$$x''(i) = x(i) \text{ for } i \in G.$$

If S'' has been chosen sufficiently small then $T(x'') \in S$ for a given S .

Furthermore, x'' is a fair (envy-free) allocation with $T(x) \leq T(x'')$,

$T(x) \neq T(x'')$. The last condition follows from the construction of x'' and the envy-free property is guaranteed by the relations: $p_m'' = p_m \succ_i p_{m*}''$, $i \in G$ and $p_{m*}'' \succeq_i p_m = p_m''$ for $i \in G$. The lemma is now completely proved. ■

Lemma 4: Under the assumptions in T1:l8: $\bar{x}$ is a fair allocation in $\mathcal{E}_2(\bar{\alpha}, n) \Rightarrow$ to any real numbers α', α'' , with $\alpha' < \bar{\alpha} < \alpha''$, there exist fair allocations x' and x'' in $\mathcal{E}_2(\alpha', n)$ and $\mathcal{E}_2(\alpha'', n)$ such that $T(x') \leq T(\bar{x}) \leq T(x'')$, $T(x') \neq T(\bar{x}) \neq T(x'')$.

Proof: Define $K = \{\alpha : \alpha \in \mathbb{R} \text{ and there exists a fair allocation } x \text{ in } \mathcal{E}_2(\alpha, n) \text{ with } \alpha' \leq \alpha \leq \alpha'' \text{ and } T(x) \leq T(\bar{x}) \text{ if } \alpha \leq \bar{\alpha} \text{ and } T(\bar{x}) \leq T(x) \text{ if } \bar{\alpha} \leq \alpha\}$. K is a subset of $\mathbb{R}$ which is bounded and, because of the continuity, of preferences, closed. K is therefore compact. Because of L3, $\alpha' = \inf K$ and $\alpha'' = \sup K$ and because K is compact $\alpha', \alpha'' \in K$. This proves the lemma. ■

Lemma 5: Under the assumptions in T1:l8: $\bar{x}$ is a fair allocation in $\mathcal{E}_2(\bar{\alpha}, n) \Rightarrow$ to any real numbers α', α'' with $\alpha' < \bar{\alpha} < \alpha''$ there exist fair allocations x' and x'' in $\mathcal{E}_2(\alpha', n)$ and $\mathcal{E}_2(\alpha'', n)$ such that $T(x') < T(\bar{x}) < T(x'')$.

Proof: 1) $\alpha' < \bar{\alpha}$.

According to L5 there is a fair allocation x^1 in $\mathcal{E}_2(\alpha^1, n)$ with $\alpha' < \alpha^1 < \bar{\alpha}$ such that $T(x^1) \leq T(\bar{x})$, $T(x^1) \neq T(\bar{x})$.

Define m_1 and G_1 according to:

$$m_1 = \{e_k : e_k \in E \text{ and } p_k^1 = \bar{p}_k\}. T(x^1) = p^1 \text{ and } T(\bar{x}) = \bar{p}.$$

$$G_1 = \{i : i \in I \text{ and } x^1(i) = (e_k, p_k^1) \text{ for } e_k \in m_1\}.$$

We now have $m_1 \subset A$, $m_1 \neq A$ and thereby also that $G_1 \subset I$, $G_1 \neq I$. Furthermore, $p_{m_1}^1 \succ_i p_{m*}^1$ for $i \in G_1$ and $p_{m*}^1 \succeq_i p_{m_1}^1$ for $i \in G_1^*$. Because of the continuity of the preferences there is a neighbourhood S^1 to p^1 such that

$$p_{m_1}^1 \succ_i p_{m*}^1 \text{ for } i \in G_1 \text{ if } p \in S^1.$$

Now consider only elements from m_1 and individuals from G_1 and repeat the

procedure once more. Then according to L4 there is for each $\varepsilon > 0$ a fair

allocation $x_{m_1}^2$ in $\mathcal{G}_2(\alpha_{m_1}^2, n^1)$ with $\alpha_{m_1}^1 - \varepsilon < \alpha_{m_1}^2 < \alpha_{m_1}^1$ such that

$T(x_{m_1}^2) \leq T(x_{m_1}^1)$, $T(x_{m_1}^2) \neq T(x_{m_1}^1)$. n^1 = the number of elements in m_1 (or G_1)

and $\alpha_{m_1}^1 = \sum_{e_k \in m_1} p_k^1$.

An allocation x^2 in $\mathcal{G}_2(\alpha^2, n)$ is now defined by:

$$x^2(i) = x_{m_1}^2(i) \text{ for } i \in G_1.$$

$$x^2(i) = x^1(i) \text{ for } i \in G_1^*.$$

$$\alpha^2 = \alpha_{m_1}^2 + \alpha^1 - \alpha_{m_1}^1.$$

For sufficiently small ε and for $T(x^2) = p^2 \in S^1$ x^2 has the following properties:

i) Is envy-free because $p_{m_1 i}^2 > p_{m_1 i}^{2*}$ for $i \in G$ ($x^2 \in S^2$) and $p_{m_1 i}^{2*} \approx p_{m_1 i}^2$ for $i \in G_1^*$ ($p_{m_1 i}^2 \leq p_{m_1 i}^1$).

ii) $\alpha' < \alpha^2 < \alpha^1 < \bar{\alpha}$ because $\alpha^1 - \varepsilon < \alpha^2$.

With x^2 as a starting point we can repeat the procedure with m_2 and G_2 defined by $m_2 = \{e_k : e_k \in E \text{ and } p_k^2 = \bar{p}_k\}$ and $G_2 = \{i : i \in I \text{ and } x^2(i) = (e_k, p_k^2) \text{ for } e_k \in m_2\}$.

In this case we also have $m_2 \subset m_1$, $m_2 \neq m_1$ and $G_2 \subset G_1$, $G_2 \neq G_1$. By further repetitions we get a sequence $I \supset G_1 \supset G_2 \dots$ where each group is strictly smaller than its predecessor and because I is finite $G_k = \emptyset$ for some finite k .

Then we have a fair allocation x^k in $\mathcal{G}_2(\alpha^k, n)$ with $\alpha' < \alpha^k < \bar{\alpha}$ and $T(x^k) < T(\bar{x})$.

By using L4 once more we get a fair allocation x' in $\mathcal{G}_2(\alpha', n)$ with $T(x') \leq T(x^k) < T(\bar{x})$ and this is the allocation required.

2) $\bar{\alpha} < \alpha''$. This case is proved in exactly the same way as case 1). L5 is now completely proved. ■

We shall now use L5 to give a proof of Theorem 1 in sec. 18.

Proof of T1:18: We shall prove T1:18 by assuming it to be true (the induction assumption) for each integer $n \leq N$, where n is the number of individuals or the number of indivisibles, and show that this assumption implies that T1:18 is true also for $n = N + 1$. In that case T1:18 must be true for any integer N , because T1:18 is trivially true for $N = 1$.

Now consider E and I with $N + 1$ elements and let $E' \subset E$, $I' \subset I$ be subsets with N elements. $E = E' \cup \{e_{N+1}\}$ and $I = I' \cup \{i_{N+1}\}$. We also consider a given amount $\bar{\alpha}$ of the divisible commodity.

Define M to be $M = \{x : x \text{ is an allocation in } \mathcal{E}_2(\bar{\alpha}, N+1) \text{ such that } x(i_{N+1}) = (e_{N+1}, p_{N+1}) \text{ and the restriction of } x \text{ to } I', x_{I'}, \text{ is a fair allocation in } \mathcal{E}_2(\bar{\alpha} - p_{N+1}, N)\}$.

According to the induction assumption $M \neq \emptyset$ for any $\bar{\alpha} \in \mathbb{R}$ and for any choice of p_{N+1} .

We put $\bar{p}_{N+1} = \sup\{p_{N+1} : x \in M, T(x) = p \text{ and } p_E \succsim_i p_{N+1} \forall i \in I\}$ and prove that $-\infty < \bar{p}_{N+1} < \infty$.

That $-\infty < \bar{p}_{N+1}$ can be proved in the following way:

The assumption in T1:18, that for some x there are allocations x^i in $\mathcal{E}_2(\alpha, N+1)$ such that $x^i(j) \sim x^i(k) \forall j, k \in I$, and T5:19 implies the existence of allocations x^i in $\mathcal{E}_2(\bar{\alpha}, N+1) \forall i \in I$ such that $x^i(j) \sim x^i(k) \forall j, k \in I$.

This implies that $p_E \succsim_i p_{N+1} \forall i \in I$ for $x \in M$ and $T(x) = p$ if $p_{N+1} < \min_{i \in I} p_{N+1}^i$, $T(x^i) = p^i$. Thus $-\infty < \bar{p}_{N+1}$.

The relation $\bar{p}_{N+1} < \infty$ is true because $\bar{p}_{N+1} \leq \max_{i \in I} p_{N+1}^i$.

Now the continuity of the preferences implies the existence of an allocation $\bar{x} \in M$ such that $\bar{x}(i_{N+1}) = (e_{N+1}, \bar{p}_{N+1})$. Furthermore, from the continuity

and L5 follows the existence of $i \in I$ such that $(e_{N+1}, \bar{p}_{N+1}) \underset{i}{\sim} \bar{x}(i)$.

About $\bar{x}$ can summarily be said that the restriction $\bar{x}_I$, to I' is a fair allocation. Individual i_{N+1} has been assigned the element $(e_{N+1}, \bar{p}_{N+1})$ and for some $i \in I$ $x(i_{N+1}) \underset{i}{\succ} x(j) \quad \forall j \in I$.

We shall now prove the existence of a fair allocation $\bar{\bar{x}}$ with $\bar{p} = T(\bar{x}) = T(\bar{\bar{x}})$ i.e. the distribution of the divisible commodity is the same in $\bar{x}$ and $\bar{\bar{x}}$ but the distribution of indivisibles among individuals may be different.

Assume that no such allocation $\bar{\bar{x}}$ exists i.e. that $\bar{\bar{x}}$ is a fair allocation with $T(\bar{x}) = T(\bar{\bar{x}})$. We prove that this assumption (A) leads to a contradiction, that $\bar{p}_{N+1}$ could not be a supremum.

A implies that the relation $\bar{p}_{N+1} \underset{i_{N+1}}{\succ} \bar{p}_k$ can not be true for all k , because in that case $\bar{x}$ would be fair. We leave individual i_{N+1} out of account and consider allocations $: I' \rightarrow E \times R$ which all have the same distribution of the divisible commodity as $\bar{x}_I$, and construct a sequence $X = \{x^k\}_{k=1}^{\infty}$ of these kinds of allocations which also have the following properties:

- i) $x^k: I' \rightarrow E \times R \quad \forall k$ and $T(x^k) = T(\bar{x}_I)$.
- ii) $x^1 = \bar{x}_I$.
- iii) $x^k(i) \neq x^k(j)$ if $i \neq j$.
- iv) $x^k(i) \neq (e_j, \bar{p}_j) \quad \forall i \in I$ for some $e_j \in E$ implies that $x^{k+1}(i) = (e_j, \bar{p}_j)$ for some $i \in I'$ with $\bar{p}_j \underset{i}{\succ} \bar{p}_r$ for $r = 1, 2, \dots, N+1$.
- v) X shall include a maximal number of different elements. (There can only be a finite number of different elements in X !)

From the construction of X immediately follows that each allocation $x^k \in X$ is envy-free.

Define $m = \{e_k : e_k \in E' \text{ and there is } x \in X \text{ with } x(i) \neq (e_k, \bar{p}_k) \quad \forall i \in I'\}$.

We know that $\underline{m} \neq \emptyset$ because $(e_{N+1}, \bar{p}_{N+1}) \underset{i}{\sim} x^1(i)$ for some $i \in I'$ according to A. Furthermore, $\underline{m} \neq E$ because for $m = E$ we get a fair allocation $\bar{\bar{x}}$ if i_{N+1}

is assigned an element in $\{(e_k, \bar{p}_k)\}_{k=1}^{N+1}$ which he prefers and for individuals in I' we choose an element $x \in X$ where $x(i) \neq \bar{x}(i_{N+1})$ for $i \in I'$ and put $\bar{x}(i) = x(i)$ for $i \in I'$.

We also have the relation $\bar{x}(i) \succ_i (e_k, \bar{p}_k) \quad \forall e_k \in m \cup \{e_{N+1}\}$ if $i \in G =$
 $= \{i : i \in I \text{ and } \bar{x}(i) = (e_k, \bar{p}_k), e_k \in m^* = E' - m\}$ because X includes a maximal number of different elements.

We can now construct an allocation $\bar{x} \in M$ ($T(\bar{x}) = \bar{p}$) such that $\bar{p}_{N+1} < \bar{p}_{N+1}$ and $\bar{p}_k \succ_i \bar{p}_{N+1} \quad \forall i \in I$ and for all $k = 1, 2, \dots, N$. This is a contradiction to the choice of $\bar{p}_{N+1}$ as a supremum and consequently the assumption A is false. Thus, there must be a fair allocation in $\mathcal{G}_2(\bar{\alpha}, N+1)$.

The construction of $\bar{x}$ remains:

Let S' and S'' be neighbourhoods to the allocations $\bar{x}_G$ and $\bar{x}_{G^*}$, $G^* = I' - G$.

According to L5 and the induction assumption we can find fair allocations

$\bar{x}_G : G \rightarrow m^* \times R$ and $\bar{x}_{G^*} : G^* \rightarrow m \times R$ such that:

- i) $\bar{x}_G \in S'$ and $\bar{x}_{G^*} \in S''$
- ii) $T(\bar{x}_G) < T(\bar{x}_G)$ and $T(\bar{x}_{G^*}) > T(\bar{x}_{G^*})$
- iii) If $T(\bar{x}) = \bar{p}$ then $\sum_{k=1}^{N+1} \bar{p}_k = \bar{\alpha}$ where

$\bar{x}(i) = \bar{x}_G(i)$ for $i \in G$, $\bar{x}(i) = \bar{x}_{G^*}(i)$ for $i \in G^*$

and $\bar{x}(i_{N+1}) = (e_{N+1}, \bar{p}_{N+1} + \varepsilon)$, $\varepsilon > 0$.

If S', S'' and ε are sufficiently small then the following relations are also true:

- i) $\bar{x}_G(i) \succ_i \bar{x}_{G^*}(j) \quad \forall i \in G \text{ and } \forall j \in G^*$
- ii) $\bar{x}_{G^*}(i) \succ_i \bar{x}_G(j) \quad \forall i \in G^* \text{ and } \forall j \in G$
- iii) $\bar{x}(i) \succ_i (e_{N+1}, \bar{p}_{N+1}) \quad \forall i \in I$.

The relations show that $\bar{x}_I$ is envy-free, $\bar{x} \in M$ and that $\bar{p}_{N+1}$ was not a

supremum (iii). The proof is now complete. ■

21. A Generalization of the Feasibility Set

In the model $\mathcal{E}_2$ we have assumed that feasible distributions (p) of the divisible good were those where $\sum p_k \leq \alpha$. In this section we shall analyse the existence of fair allocations for some other kind of feasibility sets. We shall consider continuous deformations of the feasibilities from $\mathcal{E}_2$. We shall also give a description of the set if all fair allocations in $\mathcal{E}_2(\alpha, N)$ when α varies. Now define a generalization of $\mathcal{E}_2$, which we call $\mathcal{E}_3$, in the following way:

$\mathcal{E}_3$ is defined by the same assumptions as in $\mathcal{E}_2$ except for A4:18. Thus in $\mathcal{E}_3$ we assume A1:18, A2:18, A3:18 and A4 according to:

Assumption 4: Feasible allocations are $F = \{x : x \text{ is an allocation, } f(T(x)) \leq \alpha \text{ and } f \text{ is a continuous and strictly increasing}^1 \text{ function } R^N \rightarrow R\}$.

We get A4:18 from A4 if we put $f(p) = \sum_{k=1}^N p_k$. Other examples of f are $f(p) = p \cdot q$, $q \in R^N$ and $q > 0$ or $f(p) = \sum_{k=1}^N p_k^2$.

The main theorem of this section will be a proof for the existence of fair allocations in $\mathcal{E}_3$, but for this proof we shall need some other theorems which also have an interest of their own. We start by showing a simple way of limiting the area in which fair allocation may exist in $\mathcal{E}_2(\alpha, N)$.

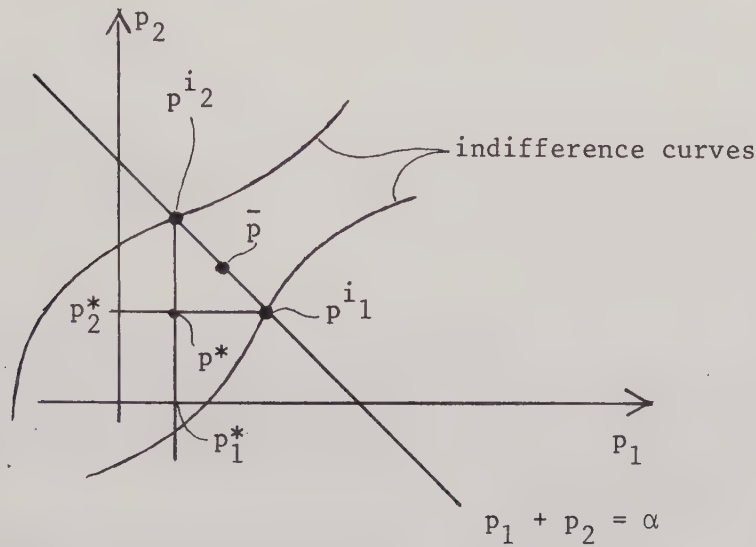
Theorem 1: If $\bar{x}$ is a fair allocation in $\mathcal{E}_2(\alpha, N)$, and x^i with $T(x^i) = \alpha$ are allocations such that $x^i(j) \sim_i x^i(k) \quad \forall i, j, k \in I$ then $\bar{p} \geq p^*$ where $p_k^* = \min_{i \in I} p_k^i$. $T(\bar{x}) = \bar{p}$ etc.

Proof: Suppose $\bar{x}(i) = (e_k, \bar{p}_k)$. If $\bar{p}_k < p_k^i$ then $\bar{p}_j < p_j^i \quad \forall j$. This follows

¹ By a strictly increasing function F we mean: $p' \geq p''$, $p' \neq p'' \Rightarrow F(p') > F(p'')$ for $p', p'' \in R^N$, $p' > p''$ if $p'_k \geq p''_k \quad \forall k$.

from the definition of p^i and the strictly monotone property of the preferences. However, $\sum_j \bar{p}_j = \sum_j p_j^* = \alpha$ so we must have $\bar{p}_k \geq p_k^i$. $\bar{p}_k \geq p_k^i \Rightarrow \bar{p}_k \geq p_k^*$. This relation must hold for each $k = 1, 2, \dots, N$ because all individuals have been assigned different elements e_k . ■

The theorem can be illustrated graphically.



T1 can be used as a sufficient criterion for the existence of positive fair allocations $\bar{x}$ in $\mathcal{G}_r(\alpha, N)$ i.e. $T(\bar{x}) \geq 0$. We just have to check that the differences x^i , $i \in I$ in $\mathcal{G}_2(\alpha, N)$ are positive i.e. $T(x^i) \geq 0 \quad \forall i \in I$.

Now consider hyperplanes $\pi(\alpha) = \{p : p \in \mathbb{R}^N \text{ and } \sum_{k=1}^N p_k = \alpha\}$ in $\mathbb{R}^N$. Let

$\{x^i\}_{i \in I}$ be a set of indifferences in $\mathcal{G}_2(\alpha, N)$ i.e. $x^i(j) \sim x^i(k) \quad \forall i, j, k \in I$.

We define a function $p^*(\alpha)$ from $\mathbb{R}$ to $\mathbb{R}^N$ according to: $p_k^*(\alpha) = \min_{i \in I} p_k^i(\alpha)$ where $p^i(\alpha) = T(x^i)$.

$p^*(\alpha)$ is a continuous, and in each coordinate, strictly increasing function because of T6:19.

Also define $\pi^*(\alpha) = \{p : p \in \pi(\alpha) \text{ and } p \geq p^*(\alpha)\}$ and $\mathcal{J} = \bigcup_{\alpha \in \mathbb{R}} \pi^*(\alpha)$.

It was shown in T1 that each fair allocation in $\mathcal{G}_2$ belongs to $\mathcal{J}$. $\mathcal{J}$ is also

a closed set.

Theorem 2: $\mathcal{J}$ is a closed subset of $\mathbb{R}^N$.

Assume the set $\mathcal{J}$ not to be closed and let $\bar{p} \in \partial \mathcal{J} - \mathcal{J}$. Consider a sequence $p^k \in \mathcal{J}$ converging to $\bar{p}$ when $k \rightarrow \infty$.

To each p^k there is exactly one hyperplane $\pi(\alpha^k)$ with $p^k \in \pi(\alpha^k)$ and for $\bar{p}$ we have a unique $\pi(\alpha)$ with $\bar{p} \in \pi(\alpha)$.

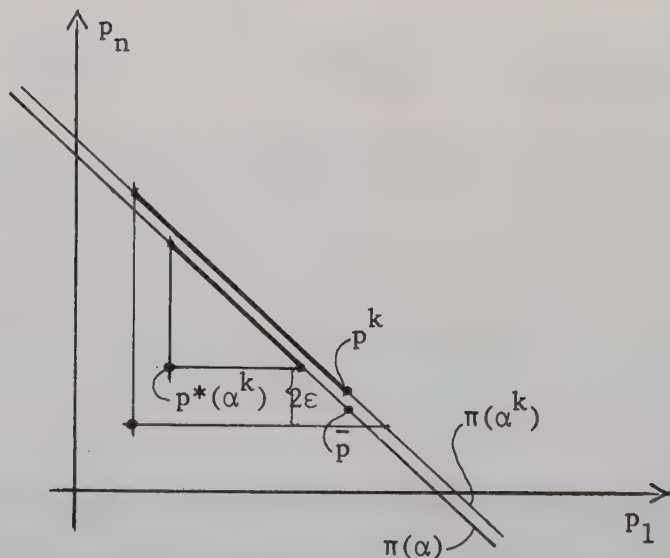
Obviously $p^k \rightarrow \bar{p} \Rightarrow \alpha^k \rightarrow \alpha$ when $k \rightarrow \infty$.

In each $\pi(\alpha^k)$ there are unique indifferences $x^i(\alpha^k) \in \pi(\alpha^k)$ for all $i \in I$ and corresponding $p^*(\alpha^k)$.

By definition of p^k we have $p^k \geq p^*(\alpha^k)$. But $\bar{p} \in \pi(\alpha)$, $\bar{p} \in \pi^*(\alpha) \Rightarrow$ there is $\varepsilon > 0$ and $\bar{p}_j$ such that $\bar{p}_j < p_j - 2\varepsilon$ for all $p \in \pi^*(\alpha)$. (Because the relation $\bar{p}_j \geq p_j^*(\alpha)$ is not true for all j !)

Because $p^k \rightarrow \bar{p}$ and $\bar{p} \notin \pi^*(\alpha)$ the relation $p_j^k < p_j - \varepsilon$ for all $p \in \pi^*(\alpha)$ must be true for all $k \geq k_0$ for some k_0 .

We now have: $p_j^k \geq p_j^*(\alpha^k) > p_j^k + \varepsilon$ for $k \geq k_0$ which is a contradiction, and the set $\mathcal{J}$ must be closed. ■



Theorem 3: $\mathcal{J}(\alpha', \alpha'') = \{p : p \in \mathcal{J} \text{ and } \alpha' \leq \sum_{k=1}^N p_k \leq \alpha''\}$ is a compact subset of R^N .

Proof: $\mathcal{J}(\alpha', \alpha'')$ is closed because it is the intersection of three closed sets. To prove the compactness we have only to check that $\mathcal{J}(\alpha', \alpha'')$ is also bounded.

We can write $\pi^*(\alpha') = \pi(\alpha') \cap K(\alpha')$ where $K(\alpha') = \{p : p \geq p^*(\alpha')\}$. We also have $\pi^*(\alpha) \subset K(\alpha')$ for $\alpha \geq \alpha'$ because $K(\alpha) \subset K(\alpha')$ if $\alpha \geq \alpha'$. It follows that $\mathcal{J}(\alpha', \alpha'') \subset K(\alpha')$. However, $\mathcal{J}(\alpha', \alpha'') \subset \{p : \sum_k p_k \leq \alpha''\} = \Psi$ which implies that $\mathcal{J}(\alpha', \alpha'') \subset K(\alpha') \cap \Psi$, and $\mathcal{J}(\alpha', \alpha'')$ is obviously bounded. ■

We can now prove an existence theorem for fair allocations in $\mathcal{E}_3$. Just as in the case with $\mathcal{E}_2$ the Pareto-optimum property of a fair allocation is only a condition that all resources have been used. We state this as a lemma before the main theorem.

Lemma: If x is an envy-free allocation in $\mathcal{E}_3$ and $f(p) = \alpha$, $T(x) = p$ then x is fair.

Proof: Because f is a strictly increasing function we can prove this lemma in exactly the same way as L1:20. ■

The lemma tells us that each fair allocation x in $\mathcal{E}_3$ is also a fair allocation in $\mathcal{E}_2(\alpha, N)$ if $T(x) = p$ and $\sum_{k=1}^N p_k = \alpha$.

Theorem 4: In $\mathcal{E}_3$ there exists a fair allocation if we can find two envy-free allocations x' and x'' , such that $f(T(x')) \leq \alpha \leq f(T(x''))$.

Proof: Define β', β'' according to $\beta' = \sum p'_k$, $\beta'' = \sum \beta''_k$ and $p' = T(x')$, $p'' = T(x'')$. $\mathcal{J}(\beta', \beta'')$ is a compact subset of R^N according to T3.

We also define $M = \{\beta : \beta = \sum p_k \text{ where } p = T(x) \text{ and } x \in \mathcal{J}(\beta', \beta''), x \text{ envy-free and } f(T(x)) \leq \alpha\}$. $M \neq \emptyset$ because $\beta' \in M$. For $\bar{\beta} = \sup M$ the relations $\beta' \leq \bar{\beta} \leq \beta''$ are true.

Now let $\{x^k\}_{k=1}^\infty$ be an infinite sequence of envy-free allocations, such that $T(x^k) \in \mathcal{J}(\beta', \beta'')$, $F(T(x^k)) \leq \alpha$ and $\beta^k \rightarrow \bar{\beta}$, $\beta^k = \sum_j p_j^k$ and $p^k = T(x^k)$.

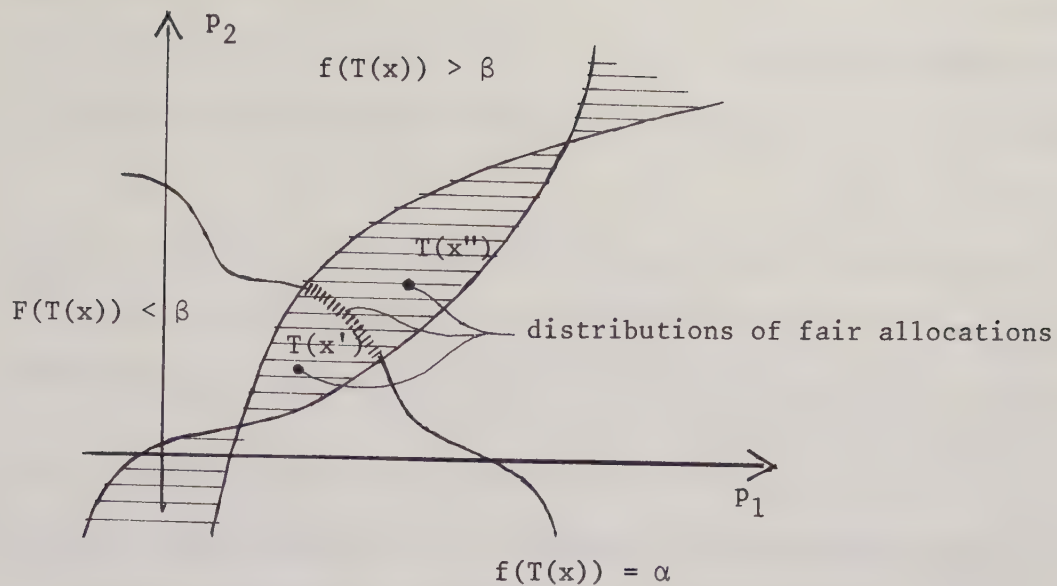
Because $\mathcal{J}(\beta', \beta'')$ is compact, there must be a convergent subsequence $\{p^{k_j}\}_{j=1}^\infty$ of $\{p^k\}$ i.e. $p^{k_j} \rightarrow \bar{p}$ for some $\bar{p}$. Furthermore, this subsequence can be chosen so that in the allocations $x^{k_j}(i)$ every individual is assigned the same element in E for all k_j . This is possible because there is a finite number N of individuals and thereby also a finite number ($N!$) of assignments. For this assignment an allocation $\bar{x}$ is defined by $\bar{x}(i) = (e_k, \bar{p}_k)$, $i \in I$ where e_k is chosen so that $x^{k_j}(i) = (e_k, p_k^{k_j})$. The allocation $\bar{x}$ has the following properties:

- i) $\bar{x}$ is envy-free because x^{k_j} are envy-free and because of the continuity of the preferences.
- ii) $f(T(\bar{x})) = \beta$ because $f(T(\bar{x})) > \alpha$ is not true according to the definition of $\bar{x}$ and $\{x^k\}$.

If $f(T(\bar{x})) < \alpha$ we can find an envy-free allocation $\bar{\bar{x}}$ where $\bar{\bar{x}} > \bar{x}$ and $f(T(\bar{\bar{x}})) < \alpha$. This follows from the fact that f is continuous and strictly increasing, and from L5:20. However, $\bar{\bar{x}} > \bar{x}$ implies that $\bar{\bar{\beta}} > \bar{\beta}$, where $\bar{\bar{\beta}} = \sum_k \bar{\bar{p}}_k$, $\bar{\bar{p}} = T(\bar{\bar{x}})$, which is a contradiction to the definition of $\bar{\beta}$ as a supremum.

Now follows that $\bar{x}$ is a fair allocation in $\mathcal{E}_3$ because according to i), $\bar{x}$ is envy-free and according to ii) and the lemma, $\bar{x}$ is Pareto-optimal. ■

The theorem can graphically be illustrated in the following way:



Corollary: If f in T4 is assumed to be an increasing but not necessarily a strictly increasing function then, under the same assumptions as in T4, there is always an envy-free allocation in $\mathcal{C}_3$.

Proof: In the proof of T4 we used the strictly increasing property of F only when we proved the efficiency property of a fair allocation. ■

22. Incomplete Preferences

In the previous sections we have assumed the whole time that all individuals are capable of ranking all elements in $E \times R$. However, in some applications of the models with indivisibles, this is too strong an assumption. If, for instance, E is a set of jobs and the divisible good assigned to a certain job represents the wage, then it may be unrealistic to assume that all kinds of jobs can be ranked. In this section we shall analyse the existence of fair allocations when a special kind of incompleteness exists in the preferences.

We define an economy $\mathcal{C}_4$ to be the same as $\mathcal{C}_2$, except for the incompleteness of preferences, which instead are assumed to fulfil a condition A1.

Assumption 1: To each individual $i \in I$ is assigned a non-empty set $m_i \subset E$ such that i has complete preferences over $m_i \times R$.

In section 19 we have analysed the properties of preferences in ξ_2 . This analysis can, of course, be transformed to ξ_4 , so we have only to substitute E for m_i . The main results are:

- i) Preferences can be described by a continuous utility function $u_i(e_k, p_k)$ for $(e_k, p_k) \in m_i \times R$.
- ii) There is a generalized indifference function $f^i : \mathcal{D}_i \rightarrow \bar{R}^N$, with the same properties as the function in T7:19, except for $\mathcal{D}_i$ being the union of intervals I_k (the range of $u_i(e_k, p_k)$ for fix p_k) for which $e_k \in m_i$ and that $f_k^i(\) = +\infty$ if $e_k \notin m_i$.

The point ii) has consequences for the definition of an envy-free allocation, because according to ii) preferences are considered as complete, but if $e_k \in m_i$ and $e_j \notin m_i$ then $(e_k, p_k) \succsim_i (e_j, p_j)$ for all finite p_k and p_j . Thus, for an envy-free allocation x we have $\forall i \in I$:

$$x(i) \succsim_i x(j) = \forall j \text{ such that } x(j) = (e_k, p_k) \text{ with } e_k \in m_i.$$

An individual can only envy individuals who have been assigned an element from E which also belongs to his own m_i .

For a proof of the existence of fair allocations in ξ_4 we shall need some complementary assumptions and lemmas.

Assumption 2: There is a bijection $\varphi : I \rightarrow E$ such that $\varphi(i) \in m_i$.

If A2 is not fulfilled, then in each allocation x , some individual i must be assigned an element $e_k \in E$ such that $e_k \notin m_i$, so x will not be envy-free.

Thus A2 is necessary for the existence of fair allocations.

Next assumption will not be necessary for the final proof, but it is a useful means of assistance. In fact $A3 \Rightarrow A2$!

Assumption 3: If $G \subset I$, $G \neq I$ and $G \neq \emptyset$ then there are more different elements in $\bigcup_{i \in G} m_i$ than the number of individuals in G .

For the existence proof we shall use one of A4, A5 or A6.

Let f^i be the generalized indifference function which describes completely the preferences $\succeq_i$ of individual $i \in I$, and assume that the definition set has no upper or lower bound.

Assumption 4: $\forall i \in I : f_k^i(t) < \infty$ if $e_k \in m_i$.

Assumption 5: $\forall i \in I : f_k^i(t) > -\infty$ if $e_k \in m_i$.

A4 and A5 can be illustrated graphically.

fig. 1

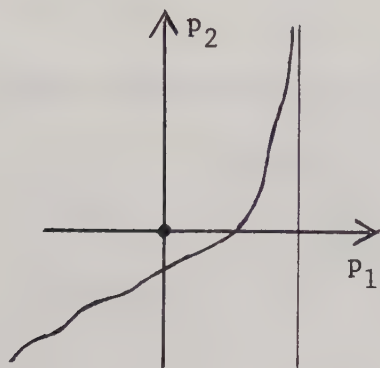


fig. 2

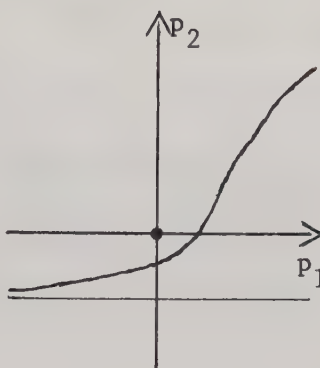
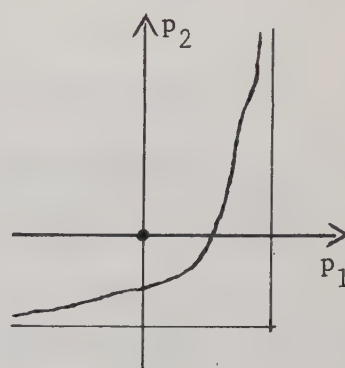
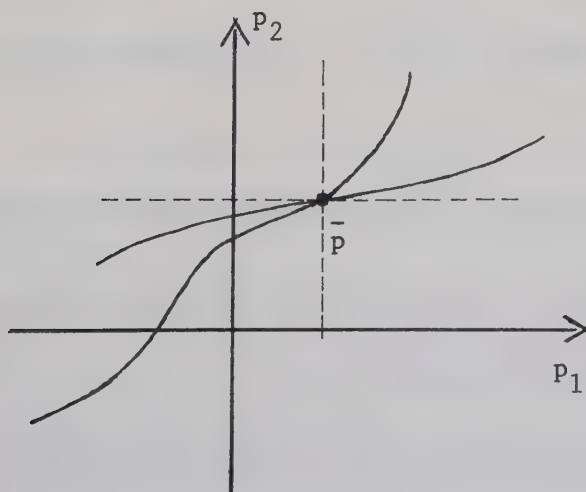


fig. 3



In fig. 1 $f^i_2(t) = \infty$ for some t , in fig. 2 $f^i_1(t) = -\infty$ for some t and fig. 3 there is t such that $f^i_2(t) = \infty$ or $f^i_1(t) = -\infty$.

Assumption 6: There is $\bar{p} \in \mathbb{R}^N$ and $t_i \in \mathbb{R}$ such that $\forall i \in I : f_k^i(t_i) = \bar{p}_k$ if $e_k \in m_i$.



Lemma 1: If x is an envy-free allocation in $\mathcal{E}_4$ and $\sum_{k=1}^n p_k = \alpha$, $T(x) = p$, then x is fair.

Proof: The lemma is the same as L1:20 and the proof of L1:20 is valid even for incomplete preferences according to A1. ■

Lemma 2: In $\mathcal{E}_4$ there exists a fair allocation if we assume A3 and A4 (or A3 and A5).

Proof: If $f^i(t)$ is the generalized indifference function of individual $i \in I$ we define a corresponding generalized indifference function $f^i(u, t)$ $u \in \mathbb{R}$ according to:

$$f_k^i(u, t) = f_k^i(t) \text{ for } k : e_k \in m_i \text{ and } f_k^i(u, t) = u + t \text{ for } k : e_k \notin m_i.$$

$f^i(u, t)$ represents (extended) preferences which for large u is an approximation of $\succsim_i$. The existence of fair allocations in $\mathcal{E}_4$ can be proved by, for each u , proving the existence of a fair allocation, and by showing that for a sufficiently large u such a fair allocation is also a fair allocation in respect to preferences $\{\succsim_i\}_{i \in I}$.

For each u we have fair allocations because preferences are complete, and the definition of $f^i(u, t)$ together with A_4 (or A_5) secure the existence of allocations x^i such that $x^i(j) \sim_i x^i(k) \quad \forall i, j, k \in I$. If e.g. A_4 is assumed then $-\infty < f^i(u, t) < +\infty$ for a sufficiently small t and we put $T(x^i) = f^i(u, t)$. The special assignment can be chosen arbitrarily.

Now consider an infinite sequence $\{u^k\}_{k=1}^{\infty}$ of real numbers such that $u^k \rightarrow \infty$ when $k \rightarrow \infty$.

To each k there is a fair allocation x^k according to the extended preferences.

If for some $k : x^k(i) \in m_i \quad \forall i \in I$ then x^k is also a fair allocation in respect to the original preferences.

In that case the theorem is proved so now consider the reverse case; there

is $i \in I$ such that $x^k(i) \notin m_i \times R$ for infinitely many k .

We assume that the sequence $\{u^k\}$ has been chosen so that for some $i \in I$ $x^k(i) \notin m_i \in R \quad \forall k$. This can always be achieved by taking a subsequence. We also assume that $x^k(i_1) = (e_1, p_1^k) \notin m_1 \times R \quad \forall k$, which can be achieved by taking a subsequence of $\{u^k\}$ and by relabeling it.

There are two cases: 1) For some $a \in R : p_1^k \leq a \quad \forall k$. 2) $p_1^k \rightarrow +\infty$ for some k .

1) $p_1^k \leq a$. We have $p_1^k = f_1^{i_1}(u^k, t^k) = t^k + u^k \leq a$ because $e_1 \notin m_1$. Furthermore, $p_j^k = t^k + u^k \leq a$ if $e_j \notin m_1$ and $t^k \rightarrow -\infty$ when $k \rightarrow \infty$. However, $t^k \rightarrow -\infty$ implies that $p_j^k \leq f_j^{i_1}(t^k)$ has a finite upper bound even for $j : e_j \in m_1$.

$t^k \rightarrow -\infty$ also implies that $f_j^{i_1}(t^k) \rightarrow -\infty$ for some j such that $e_j \in m_1$ i.e. $p_j^k \rightarrow -\infty$ because $p_j^k \leq f_j^{i_1}(t^k)$.

Thus the situation is that p_j^k have an upper bound $\forall j, \forall k$, and for some j $p_j^k \rightarrow -\infty$. This is a contradiction to $\sum_{j=1}^N p_j^k = \alpha$.

2) $p_1^k \rightarrow +\infty$ when $k \rightarrow +\infty$. (p_1^k may be a subsequence of the original sequence.)

If A4 is assumed we have $t_i^k \rightarrow +\infty$, where t_i^k is the ordinal utility

level (the argument in the extended generalized indifference function) of

$i \in G_1$ in the allocation x^k . G_1 is the group $G_1 = \{i : e_1 \in m_i\}$. (For these individuals $p_1^k \leq f_1^i(t_i^k)$!) However, $t_i^k \rightarrow \infty \Rightarrow p_j^k \rightarrow \infty$ where $x^k(i) = (e_j, p_j^k)$ if $i \in G_1$.

Because of A3 there is $i \notin G_1$ such that $x^k(j) \in m_i \times R$ for some $j \in G_1$. In that case A4 implies once more that the utility level of i $t_i^k \rightarrow \infty$.

The procedure can be repeated and we get $t_i^k \rightarrow \infty \quad \forall i \in I$ and thus

$p_j^k \rightarrow \infty \quad \forall j$. This is a contradiction to $\sum_j p_j^k = \alpha$.

If A5 is assumed instead of A4 we can argue in the same way. First

$p_1^k \rightarrow \infty \Rightarrow \exists p_j^k \rightarrow -\infty$ when $k \rightarrow \infty$. With this starting point we can prove that, in the same way as in the above, $p_j^k \rightarrow -\infty \quad \forall j$, which is a contradiction. ■

Lemma 3: To each fair allocation $\bar{x}$ in $\mathcal{G}_4$ and to each $\alpha', \alpha'' \in R$, such that $\alpha' \leq \bar{\alpha} \leq \alpha''$ and where $\bar{\alpha} = \sum \bar{p}_k$, $p = T(\bar{x})$ etc., there are fair allocations x' and x'' in $\mathcal{G}_4$ such that $T(x') \leq T(\bar{x}) \leq T(x'')$ if we assume A3 and A4 (or A3 and A5).

Proof: $f^i(t)$ and $f^i(u, t)$ are the generalized and extended generalized in-difference functions of $i \in I$ respectively. $f^i(u, t)$ is defined as in L2. For a sufficiently large u_0 and $u \geq u_0$ we have:

$$\bar{x}(i) = (e_k, \bar{p}_k) \text{ and } \bar{p}_k = f_k^i(u, t) = f_k^i(t) < u + t \quad \forall k, \forall i \in I.$$

This means that for every $u \geq u_0$ each individual $i \in I$ consider $x(i)$ at least as good as $x(j)$ even in respect to extended preferences and $x(i) \in m_i \times R$.

Consider the sequence $\{u^k\}$ from L2. For each k there are fair allocations ${}^k x'$ and ${}^k x''$ in $\mathcal{G}_2(\alpha', N)$ and $\mathcal{G}_2(\alpha'', N)$ according to L5:20, such that ${}^k p' = T({}^k x') \leq T(\bar{x}) \leq T({}^k x'') = {}^k p''$.

According to the proof of L3 at least one of the coordinates of ${}^k p'$ and ${}^k p''$ will increase to infinity when $k \rightarrow \infty$ if ${}^k x'$ or ${}^k x''$ are not fair allocations. This is a contradiction to ${}^k p' \leq \bar{p} \leq {}^k p''$, $\sum_j {}^k p'_j = \alpha'$ and $\sum_j {}^k p''_j = \alpha''$. ■

Lemma 4: For each sequence $\{\alpha^k\}_{k=1}^\infty$, $\alpha^k \rightarrow \infty$ or $\alpha^k \rightarrow -\infty$ and sequence $\{x^k\}_{k=1}^\infty$ of fair allocations in $\mathcal{G}_4$ with $\sum p_j^k = \alpha^k$, $p^k = T(x^k)$ we have:

- 1) $\alpha^k \rightarrow -\infty \Rightarrow p_j^k \rightarrow -\infty$ for each j if A3 and A5 are assumed.
- 2) $\alpha^k \rightarrow +\infty \Rightarrow p_j^k \rightarrow +\infty$ for each j if A3 and A4 are assumed.

Proof: 1) Suppose that the lemma is wrong i.e. that there is a real number a and a j , such that $p_j^k \geq a \quad \forall k$.

For individuals $i : e_j \in m_i$ the relation $t_k^i \geq b$ must be true for some $b \in R$, $f_r^i(t_k^i) = p_r^k$ and $x^k(i) = (e_r, p_r^k)$.

From A2 follows (compare with the proof of L2) that the sequence $\{t_k^i\}_k$ has a

finite lower bound $\forall i \in I$.

However, $\alpha^k \rightarrow -\infty \Rightarrow p_r^k \rightarrow -\infty$ for some r and this implies that $t_k^i \rightarrow -\infty$ for some i , which contradicts that the sequence $\{t_k^i\}_k$ is bounded for all $i \in I$.

The case 2) is proved in the same way. ■

Lemma 5: Under the assumption A2 there is $G \subset I$ such that:

- 1) $\bigcup_{i \in G} m_i = m_G$ has the same number of different elements as the number of individuals in G .
- 2) $G' \subset G$, $G' \neq G$ and $G' \neq \emptyset$ implies that $m_{G'}$ has more different elements than G' .
- 3) There is a bijection $\varphi : G \rightarrow m_G$ such that $\varphi(i) \in m_i$.

Proof: Consider nonempty groups $G_1 \subset I$. If m_{G_1} has more elements than G_1 for every choice of $G_1 \neq I$ we can put $G = I$. Otherwise there is G_1 such that m_{G_1} has at the most as many elements as G_1 . Then, according to A2, there must be the same number of elements in m_{G_1} as in G_1 .

Now repeat the procedure and we will get $G_2 \subset G_1$, $G_3 \subset G_2$ and so on, where each group is included strictly in its predecessor. G_k and m_{G_k} must contain at least one element. Consequently there is a smallest group G_k . If we put $G = G_k$ then G have the properties desired. ■

The main theorem can now be proved.

Theorem 1: In $\mathcal{C}_4$ there exists a fair allocation if we assume A2 and A4 (or A2 and A5).

Proof: Define a partition $\{G_j\}_{j=1}^r$ of I , i.e. $G_j \cap G_k = \emptyset$ if $j \neq k$ and

$\bigcup_j G_j = I$, such that:

- 1) $i \in G_j \Rightarrow m_i \cap m_{G_k} = \emptyset$ for $k > j$.
- 2) $m_{G_j} = \bigcup_{k < j} m_{G_k}$ satisfies conditions 1), 2) and 3) in L5.

Such a partition exists because we can choose G_1 according to 1), 2) and 3)

in $L5$ and G_2 and m_{G_2} from $I - G_1$ and from $E - m_{G_1}$ respectively, according to 1), 2) and 3) in $L5$. In the same way G_3 and m_{G_3} can be chosen from $I - G_1 - G_2$ and from $E - m_{G_1} - m_{G_2}$ respectively. The procedure can be repeated until we have got the desired sequence.

Now consider G_r , $m_{G_r} = \bigcup_{j < r} m_{G_j}$ and an arbitrary $\alpha_r \in R$. For this group of individuals the conditions for $L2$ are satisfied and there is a fair allocation x_r .

Now consider instead G_{r-1} and $m_{G_{r-1}} = \bigcup_{j < r-1} m_{G_j}$ and choose $\alpha_{r-1} \in R$, so that there is a fair allocation for this group which can not be envied by individuals in G_r with allocation x_r . Such an α_{r-1} exists according to $L4$. Individuals in G_{r-1} can never envy individuals in G_r which have been assigned elements in x_r , because G_{r-1} does not have preferences over elements in x_r .

Repeat the procedure and choose fair allocations for the groups

$G_r, G_{r-1}, \dots, G_1$ in the same way. If $\sum_{k=1}^r \alpha_k = \alpha$ we are ready. When $\sum \alpha_k > \alpha$ we can decrease α_1 , so that $\sum \alpha_k = \alpha$. According to $L3$ this can be done by decreasing the allocation x_1 i.e. the envy-free property is preserved. If $\sum \alpha_k < \alpha$ we can in the same way increase α_r so that $\sum \alpha_k = \alpha$, and according to $L3$ this can be done by increasing x_r at which the envy-free property remains.

We have now proved the existence of an envy-free allocation $x(i) = x_k(i)$ for $i \in G_k$ with $\sum_{k=1}^N p_k = \alpha$, $T(x) = p$ and according to $L1$ x is also fair. ■

In $T1$ we have assumed that the indifference-curves of all individuals "pass through infinity" in all coordinates. It is not essential that the point is situated in infinity, but what we have really used in the proof for $L2$ is that there is one for all individuals common point of indifferences.

Theorem 2: In $\mathcal{E}_4$ there exists a fair allocation if we assume $A2$ and $A6$.

Proof: It is only in the proof of $L2$ that we have used $A4$ or $A5$, so if $L2$ can

be proved if we assume A6 instead of A4 the rest of the proof is also done.

To prove L2 under assumption A6 instead of A4 we have only to change the way of extending the generalized indifference function. Instead of $f_k^i(u, t) = u + t$ for $k : e_k \notin m_i$ we define $f_k^i(u, t) = u(t - t_i) + \bar{p}_k$ for $k : e_k \notin m_i$.

We also have to use the fact that if x is a fair allocation in respect to extended generalized preferences then $T(x) \geq \bar{p}$ or $T(x) \leq \bar{p}$. (T1:21)

For the rest of the proof we can use the same arguments as in the proof of L2, but the infinity will in some cases be exchanged for the point $\bar{p}$. We omit the details. ■

Assumptions A4 and A5 are in a certain sense the limit of A6; we let the coordinates approach infinity. Not every coordinate, however, need in principle have an infinite limit, some may be finite and others infinite i.e. $\bar{p}_k$ is finite for some k and $\bar{p}_k$ infinite for others.

A corresponding theorem to T1 and T2 is, of course, possible to prove even in those cases. We only have to make the extension of preferences in a suitable way.

23. A Model with Several Variable Characteristics

The indivisibles considered in $\mathcal{G}_2$ in sec. 18 were characterized by two coordinates $a = (e, p) \in E \times R = A$ where p could be considered a divisible part of the good a . If e.g. a is a job, p could be the wage, but in many cases several of the characteristics of a could be more or less varied. If a is still a job then, besides the wage, even the working-hours, the distance between the place of work and the home of the worker, the degree of dirtiness, and so on, could in principle be changed within certain limits. We shall in this section consider a model which take these aspects into account. In principle an element $a \in A$ will be completely defined by a point in the set C of characteristics¹ and by a subset of C in which the characteristics can

be varied.

We define an economy $\mathcal{E}_5$ by starting with $\mathcal{E}$ and adding the following assumptions (A1, A2, A3, A4 and A5).

Assumption 1: Each element $a \in A$ is completely defined by a subset $m(a, q)$ of C . q is a real number.

Assumption 2: $C \subset \mathbb{R}^n$ i.e. $a \in A$ can be described by n properties. Furthermore, C is a topological space with the induced topology from $\mathbb{R}^n$. $\mathbb{R}^n$ has the usual topology.

Assumption 3: The preference relation $\succsim_i$ of individual $i \in I$ is a complete and continuous pre-ordering over C .

The definition of continuity is the usual one i.e. the same that was used in A3:18.

Assumption 4: Feasible allocations are $F = \{x : x(i) \in m(\varphi(i), q(\varphi(i)))$

$\forall i \in I$ for some bijection (assignment function) $\varphi : I \rightarrow A$ and

$$\sum_i q(\varphi(i)) \leq \alpha\}$$

We will call a function $q : A \rightarrow \mathbb{R}$ resource distribution. If A is a set of cars, then e.g. $q(a)$ may be the production cost of a , $q(a)$ is also a measure of the set $m(a, q(a))$.

Assumption 5: The set $m(a, q)$ is for each pair (a, q) compact and strictly increasing in q in such a way that:

$q' > q'' \Rightarrow$ to each individual $i \in I$ and each $a \in A$ there is an element $x(i) \in m(a, q')$ such that $x(i)$ is strictly better than any element in $m(a, q'')$. Furthermore, the relation $q \rightarrow m(a, q)$ is supposed to be continuous for fixed a in the following sense : $q^k \rightarrow \bar{q} \Rightarrow d(m(a, q^k), m(a, \bar{q})) \rightarrow 0$ where

1 The use of characteristics can be found in Lancaster (1966).

d is the Hausdorff distance between the two sets.¹

We are now interested in the existence of fair allocations in $\mathcal{E}_5$ and we shall prove this by first proving the existence of a fair equilibrium allocation.

Definition 1: If we can assign to each individual $i \in I$ a set $S_i \subset C$ of choices, then x is an equilibrium allocation in $\mathcal{E}_5$ in respect to $\{S_i\}_i$ if x is feasible and $\forall i \in I : x(i) \succeq_i y \quad \forall y \in S_i$.

Definition 2: An allocation x is a fair equilibrium allocation in $\mathcal{E}_5$ in respect to $\{S_i\}_i$ if x is feasible and $\forall i \in I : x(i) \succeq_i y \quad \forall y \in \bigcup_{i \in I} S_i$.

The concept equilibrium allocation is in a certain sense an extension of the concept price equilibrium allocation, because x is a price equilibrium allocation if there is a price vector p and a set of incomes $\{I_i\}_{i \in I}$ such that

$$\forall i : x(i) \succeq_i y \quad \forall y \in \{y : y \geq 0, py \leq I_i\} = S_i.$$

The sets S_i are in this case the budget sets.

A fair equilibrium allocation is an extension of the concept income-fair allocation.² x is income-fair if x is a price equilibrium allocation, where the budgetsets of all individuals are equal i.e. $S_i = S_j \quad \forall i, j \in I$. It is easy to see that if preferences are strictly monotone and S_i are budgetsets (one common price vector p but different incomes I_i) then every fair equilibrium allocation is also an income-fair allocation. The reverse is also true i.e. that an income-fair allocation is a fair equilibrium allocation.

We shall now let S_i be the sets $m(a, q)$ and can then prove that fair equilibrium allocation is a stronger concept than fair allocation.

1 For non-empty, closed sets A and B are $d(A, B) = \max(\sup_{x \in A} \inf_{y \in B} d(x, y), \sup_{x \in B} \inf_{y \in A} d(x, y))$.

2 See e.g. Pazner (1975).

Theorem 1: If for some resource distribution q with $\sum_{a \in A} q(a) = \alpha$ and assign-

ment function $\varphi : I \rightarrow A$ x is a fair equilibrium allocation with $S_i = m(\varphi(i), q(\varphi(i)))$ then x is also a fair allocation.

Proof: The envy-free property is obvious in all fair equilibrium allocations, so we only have to check the Pareto-optimum property of a fair allocation.

Now let x' be any feasible allocation i.e. for some resource distribution q' with $\sum_{a \in A} q'(a) \leq \alpha$ and some assignment function $\varphi' : x'(i) \in m(\varphi'(i), q'(\varphi'(i)))$.

We have two cases: 1) $q = q'$ 2) $q \neq q'$.

1) In this case $x(i) \succeq_i x'(i) \quad \forall i \in I$.

2) $q \neq q' \Rightarrow \exists i \in I ; q'(\varphi'(i)) < q(\varphi(i))$ because $\sum q(a) = \alpha$, However, in that case $x(i) \succ_i x'(i)$ because according to A5 there is $y \in m(\varphi'(i), q(\varphi'(i)))$ such that $y \succ_i z$ if $z \in m(\varphi'(i), q'(\varphi'(i)))$ and thus $x(i) \succeq_i y \succ_i x'(i)$ if $x'(i) = z$.

1) and 2) show that x is Pareto-optimal. ■

We can now prove the existence of fair allocations by proving the existence of fair equilibrium allocations according to T1.

Preferences over C induce preferences over a set of pairs $(a, q) \in A \times R$ i.e. q is a measure of the resource assigned to the element $a \in A$. We denote this set $Q = A \times R$. Preferences over Q are induced from preferences over C according to:

Definition 3: For $i \in I$ a preference relation $\succeq_i$ over Q is defined according to:

$(a, q) \succeq_i (a', q')$ if $\exists y \in m(a, q)$ such that $y \succeq_i z \quad \forall z \in m(a', q')$.

If we consider Q as a partially ordered metric space, in the same way as $E \times R$ was (A2:18), then preferences $\succeq_i$ over Q are of exactly the same kind as those in $\mathcal{C}_2$ (A3:18).

Theorem 2: Preferences $\succsim_i$ of individual $i \in I$ is a continuous, strictly monotone and complete pre-ordering over Q .

Proof: The reflexivity and transitivity of the relation $\succsim_i$ over C are directly transformed to the relation $\succsim_i$ over Q which is, consequently, a pre-ordering.

The completeness follows from the fact (A5) that the sets $m(a, q)$ are compact and $\succsim_i$ over C is complete and continuous. We only have to compare optimal elements from the sets $m(a, q)$. Optimal elements always exist when the sets $m(a, q)$ are compact and the relation $\succsim_i$ over C is complete and continuous.

The strictly monotone property follows in the same way. (A5!)

We now prove the continuity property. To simplify the notation we write $\succsim$ instead of $\succsim_i$. For a given $(\bar{a}, \bar{q}) \in Q$ we consider the sets

$$M = \{(a, q) : (a, q) \in Q \text{ and } (a, q) \succsim (\bar{a}, \bar{q})\} \text{ and}$$

$$L = \{(a, q) : (a, q) \in Q \text{ and } (\bar{a}, \bar{q}) \succsim (a, q)\}.$$

We have to prove that M and L are closed sets.

The case M : Suppose $(a^k, q^k) \in M$ for $k=1, 2, \dots$ and $(a^k, q^k) \rightarrow (a^*, q^*)$ when $k \rightarrow \infty$. We can assume that $a^k = a^* \quad \forall k$ because this must be true for $k \geq k_0$ for some finite k_0 .

Let $\bar{x} \in m(\bar{a}, \bar{q})$ be an optimal element in respect to preferences over C . Such an element exists because of the compactness of $m(\bar{a}, \bar{q})$ and the continuity of the preferences. Denote by $S = \{x : x \in C \text{ and } x \succsim \bar{x}\}$.

Because $(a^*, q^k) \in M$ we also have $m(a^*, q^k) \cap S \neq \emptyset \quad \forall k$.

Let $x^k \in m(a^*, q^k) \cap S$.

We have two cases: 1) $q^* \leq q^k$ for infinitely many k or 2) $q^* > q^k$ for infinitely many k .

1) There is a decreasing subsequence $q^{k_j} \searrow q^*$ which also implies that the

sequence of sets $m(a^*, q^{k_j}) \cap S$ is increasing and every element x^{k_j} belongs to the compact $m(a^*, q^{k_1}) \cap S$. Thus the sequence $\{x^{k_j}\}$ has a limit point x^* which belongs to all $m(a^*, q^{k_j}) \cap S$. However, $d(m(a^*, q^*), x^*) = 0$ since the Hausdorff distance between $m(a^*, q^*)$ and $m(a^*, q^{k_j})$ approaches zero. Furthermore, $x^* \in m(a^*, q^*)$ because this set is compact and hence $m(a^*, q^*) \cap S \neq \emptyset$. This means that $(a^*, q^*) \succeq (\bar{a}, \bar{q})$.

2) In this case there is an increasing subsequence $q^{k_j} \nearrow q^*$. However, $m(a^*, q^{k_j}) \subset m(a^*, q^*)$ in that case, and consequently $m(a^*, q^*) \cap S \neq \emptyset$ and $(a^*, q^*) \succeq (\bar{a}, \bar{q})$.

The case m: The proof in this case is exactly the same as above. The theorem is proved completely. ■

T2 shows that preferences over Q are exactly the same as preferences over $E \times R$ in sec. 18. This also means that the existence theorem from that section can be used. If we by a resource allocation in $\mathcal{O}_5$ mean a function $\xi : I \rightarrow Q$ such that $\xi(i) = q(\varphi(i))$, where q is a resource distribution and φ an assignment function $\varphi : I \rightarrow A$, then the corresponding theorem to T1:18 is:

Theorem 3: In $\mathcal{O}_5$ there exists a fair resource allocation ξ if for each individual $i \in I$ there exists a resource allocation $\xi^i = q^i \circ \varphi^i$ such that

$$\sum_{a \in A} q^i(a) = \alpha \text{ and } \xi^i(j) \sim_i \xi^i(k) \quad \forall j, k \in I.$$

Proof: Follows from T1:18.

The existence of fair equilibrium allocations and fair allocations now follows immediately.

Theorem 4: Under the same assumptions as in T3 there is a fair equilibrium allocation in $\mathcal{O}_5$ in respect to the sets $S_i = m(\varphi(i), q(\varphi(i)))$ $i \in I$ for some resource allocation $q \circ \varphi$.

Proof: According to T3 there is a fair resource allocation $\bar{\xi} = \bar{q} \circ \bar{\varphi}$ where

$\bar{q}$ is a feasible resource distribution and $\bar{\varphi}$ an assignment function.

Assign to each individual $i \in I$ the set $S_i = m(\bar{\varphi}(i), \bar{q}(\bar{\varphi}(i)))$ and an optimal element $\bar{x}(i) \in S_i$ in respect to preferences $\underset{i}{\succsim}$ over C .

Such a one always exists because of the compactness of S_i and the continuity of the preferences.

Now according to the definition of the preferences over Q and to the fairness of $\bar{x}$ $\bar{x}(i)$ is at least as good as any element in $\bigcup_{j \in I} S_j$ in respect to the preferences of i . Thus the allocation $\bar{x} : I \rightarrow C$ is an equilibrium allocation. ■

Theorem 5: Under the same assumptions as in T3 there is a fair allocation in $\mathcal{E}_5$.

Proof: Follows from T1 and T4. ■

We conclude this section by characterizing a fair equilibrium allocation according to T4 in marginal terms. This is of special interest, because the fact that C is not assumed to be a topological vector space, as the commodity space in micro economic theory usually is, implies some differences in the price characterizing of a Pareto-optimal allocation compared with the usual theory.

A fundamental theorem in micro economic theory says that each Pareto-optimal allocation can be associated with a price-vector, which also marginally defines equilibrium prices. Another way to express this is that in a Pareto-optimal allocation all individual indifference surfaces can be approximated locally with hyperplanes which all have a common normal, the price vector.

To illustrate the situation in our case, where C is not necessarily a vector space we make some further assumptions in the model $\mathcal{E}_5$. We start with a fair equilibrium allocation $\bar{x}$ and assume that:

1) Preferences $\underset{i}{\succsim}$, $i \in I$ over C are strictly monotone and can be represented

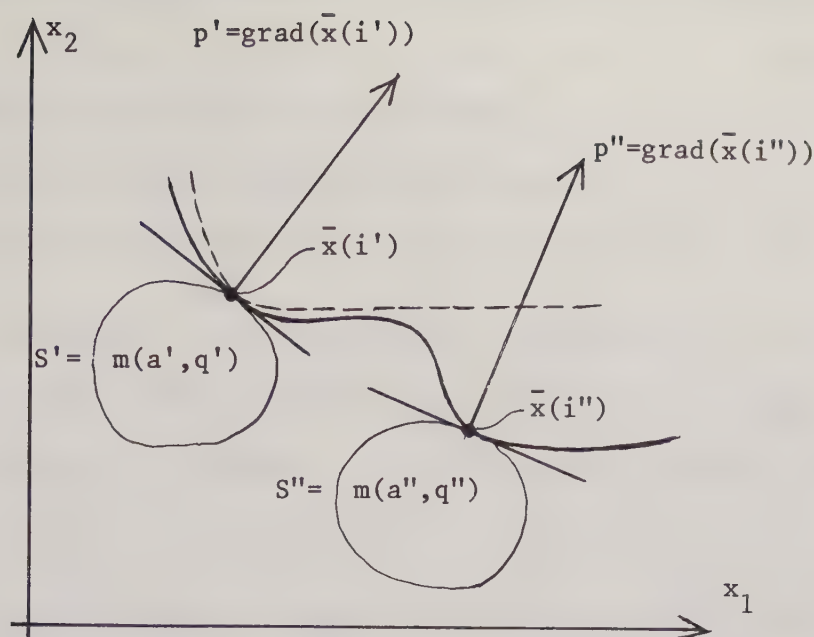
by an ordinal utility function $u_i : C \rightarrow \mathbb{R}$ which is differentiable in $\bar{x}(j) \quad \forall j \in I$.

2) If $S_i = m(\varphi(i), q(\varphi(i)))$ then the border ∂S_i of S_i is given by a relation $T_i(x) = 0$, $x \in C$ in a neighbourhood of $\bar{x}(i)$. T_i is also assumed to be differentiable in $\bar{x}(i)$. ($T_i(\bar{x}(i)) = 0!$)

The assumptions made are sufficient to give a price characterizing of a fair equilibrium allocation. A direct consequence is the existence of a set $\{p^a\}_{a \in A}$ of price vectors, one vector p^a to each element $a \in A$. p^a is of course defined by $p^a = \text{grad } u_i(\bar{x}(i)) = \text{grad } T_i(\bar{x}(i))$ if $a = \varphi(i)$. A consequence of the optimality property of $\bar{x}$ is that the indifference surface $\{x : x \in C \text{ and } u_i(x) = u_i(\bar{x}(i))\}$ and the sets S_i have a common normal p^a in the point $\bar{x}(i)$. These prices p^a express partly in what relations the individual i , who is assigned the element a , is willing to make marginal changes from $\bar{x}(i)$ with a maintained utility level, and partly what marginal efficient changes from $\bar{x}(i)$ are feasible.

However, p^a has one additional important property. If marginal changes from $\bar{x}(i)$ in the "budget set" $\{x : p^a \cdot x = p^a \cdot \bar{x}(i), x \in C\}$ are made, then the envy-free property of the allocation $\bar{x}$ remains. To prove this we can separate two cases: 1) $\bar{x}(j) \succ_j \bar{x}(i)$ for $i, j \in I$ or 2) $\bar{x}(j) \sim_j \bar{x}(i)$ for $i, j \in I$. In the first case the envy-free property remains because of the continuity property of preferences, and in the second case $p^a = \text{grad } u_i(\bar{x}(i)) = \text{grad } u_j(\bar{x}(i)) = \text{grad } T_i(\bar{x}(i))$ because $\bar{x}(i)$ maximizes $u_j(x)$ in S_i .

We can illustrate this in a figure:



Individuals i' and i'' have been assigned the elements a' and a'' respectively. In this case we have $\bar{x}(i') \sim_{i''} \bar{x}(i'')$ and $\bar{x}(i') \succ_{i'} \bar{x}(i'')$. p' and p'' are the marginal relative values of the characteristics x_1 and x_2 in the fair equilibrium allocation $\bar{x}$.

In this case i' and i'' agree on the marginal relative value p' of the characteristics of a' . If marginal changes according to prices p'' are made individual i' has no objections, because he will in any case prefer his own bundle $\bar{x}(i')$ even if he does not consider p'' as the correct relative values of the characteristics of a'' .

Summarily we can say that the fair equilibrium allocation $\bar{x}$, which is also a Pareto-optimum, can be associated with a set $\{p^a\}_{a \in A}$ of marginal equilibrium prices of the characteristics. If the set C of characteristics had been a vector space, then we could have chosen all p^a equal. Furthermore, the envy-free property of $\bar{x}$ is stable for marginal changes of $\bar{x}$ in the hyperplanes through the points $\bar{x}(i)$ with the normals p^a , $a = \varphi(i)$. For this property to exist it is essential that $\bar{x}$ is not only a fair allocation, but the stronger concept, fair equilibrium allocation.

24. A Comparison between Fair Distribution of Indivisibles and a Linear Assignment Problem

We shall conclude this chapter by comparing our model $\mathcal{E}_2$ and the existence theorem T1:18 with a similar but not so general result, which can be found in Gale (1960). Gale's result is a consequence of integral linear programming.

The following problem is considered in Gale. There are n individuals $I_1, \dots, I_n$ and n houses $H_1, \dots, H_n$ which are available for sale. Each individual demands at the most one house, and which one is depending on the sale price π_j of the house H_j .

The problem is to find a non negative price vector $p = (\pi_1, \pi_2, \dots, \pi_n)$ such that demand and supply are balanced, i.e. p is an equilibrium price vector.

To each individual I_i and house H_j we can assign the value $\alpha_{ij} \geq 0$ and I_i demands house H_j if and only if $\alpha_{ij} - \pi_j \geq 0$ and $\alpha_{ij} - \pi_j$ is a maximum when j is varied. $\alpha_{ij} - \pi_j$ measures the increase in value for I_i from the purchase of H_j .

Gale proves the existence of an equilibrium price vector p and for that uses the theory of integer linear programming.

We shall now compare Gale's model with $\mathcal{E}_2$, and also give a proof for the existence of an equilibrium price vector using the theory in this chapter.

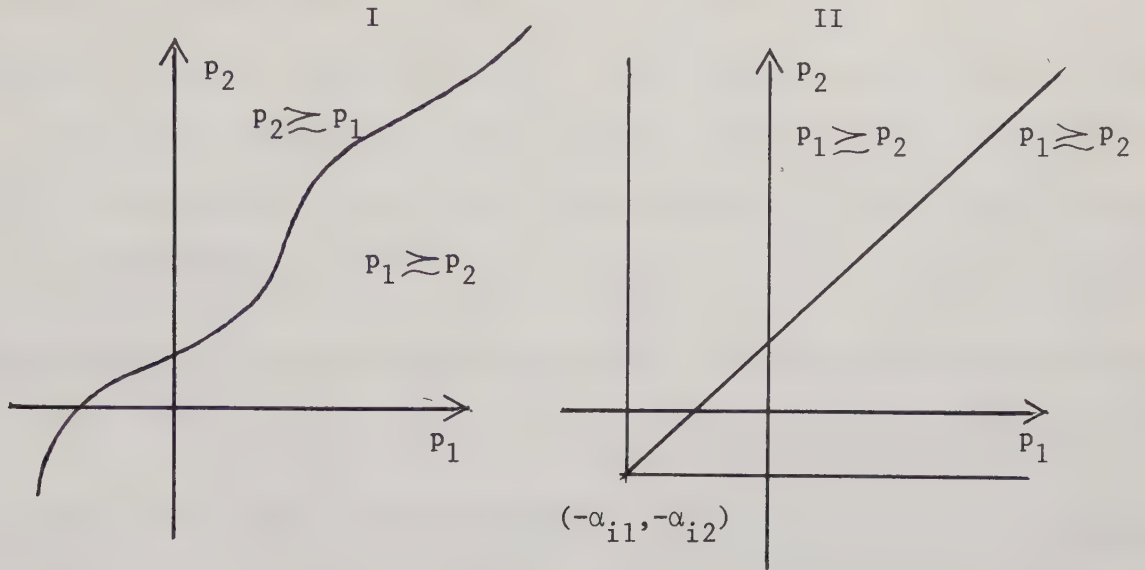
First consider the preferences of the individuals $\{I_i\}_{i=1}^n$.

It is easy to see that each individual I_i has incomplete preferences $\succsim_i$ over a set $H = \{(H_j, \pi_j) : H_j \text{ is a house, and } \pi_j \in \mathbb{R} \text{ is the sale price of } H_j\}$ which can be described by an ordinal utility function $u_i^*(H_j, \pi_j) = \alpha_{ij} - \pi_j$.

Preferences are defined in the subset of H where $u_i^* \geq 0$.

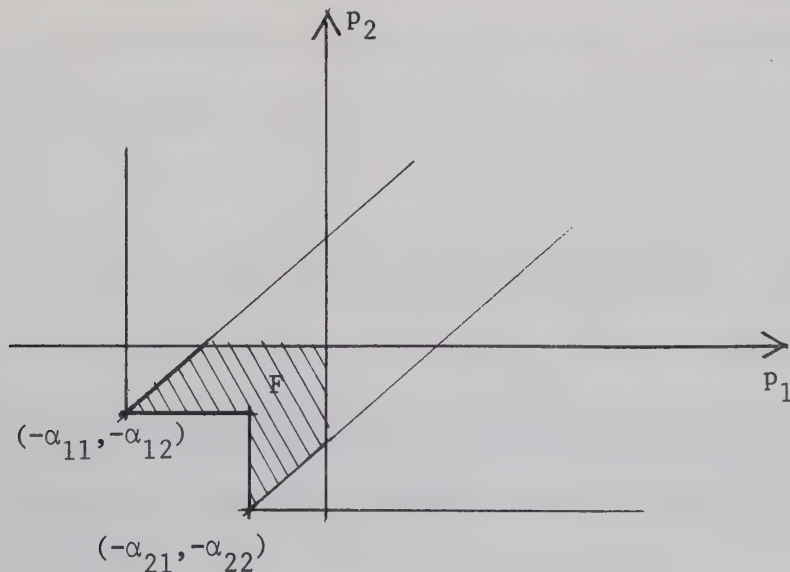
If we define $p_j = -\pi_j$ and $u_i(H_j, p_j) = u_i^*(H_j, -\pi_j) = \alpha_{ij} + p_j$, then we have

preferences which are exactly the same as those in sec. 19, but not so general. In this case u_i is a linear function of p_j , $j = 1, 2, \dots, n$, but in sec. 19 we merely assumed u_i to be a strictly increasing function of p_j . The difference is easy to illustrate in a figure.



In the first case (I) the preferences are defined in the whole R^2 and the indifference function is an increasing function. In case two (II) we have Gale's case, the preferences are defined in the cone $(p_1, p_2) \geq (-\alpha_{i1}, -\alpha_{i2})$, and the indifference function is a straight line parallel with the vector $(1, 1)$ starting at the point $(-\alpha_{i1}, -\alpha_{i2})$.

In the case of $n = 2$ the existence of an equilibrium price vector can also be illustrated in a figure in Gale's case.



All vectors in the lined area (F) are equilibrium prices which also exist according to Gale.

An allocation $x : \{I_i\}_{i=1}^n \rightarrow \{(H_j, p_j)\}_{j=1}^n$ where p is a price equilibrium vector is not exactly the same as a fair allocation, because x fulfils the envy-free condition but not an efficiency condition. In Gale's model x is efficient because besides the buyers $I_1, \dots, I_n$, there is also a number of sellers of the houses. If the interests of both these categories are taken into account, then a price equilibrium vector also defines a Pareto-optimum if we assume the sellers to desire as high prices π_j (low p_j) as possible.

Now let us see how the existence of a price equilibrium vector can be proved using the theory in this chapter.

Theorem: For any set $\{\alpha_{ij}\}_{i,j=1}^n$ of nonnegative values there exists a non-negative equilibrium price vector $(\pi_1, \pi_2, \dots, \pi_n)$.

Proof: First substitute π_j with $-p_j$ so that we can use u_i as ordinal utility functions. The preferences corresponding to u_i are in Gale's model only defined in the subset of R^n where $u_i \geq 0$. We now assume that $u_i(H_j, p_j) = \alpha_{ij} + p_j$ is defined for all real p_j , so that preferences will be complete.

Now consider as feasible distributions of the "divisible good" the set $F = \{p : p \in R^n, f(p) \leq 0 \text{ and } f(p) = \max_j p_j\}$.

It is easy to check that there are two envy-free allocations x', x'' with $T(x') = p'$, $T(x'') = p''$ and $f(p') \geq 0$, $f(p'') \leq 0$. This follows from T1:18 and T1:21.

From the corollary of T4:21 follows the existence of an envy-free allocation $\bar{x}$ with $T(\bar{x}) = \bar{p} \in F$.

To prove that $\bar{p}$ is an equilibrium price vector there only remains to prove that $\bar{p}_j \geq -\alpha_{ij}$ if house H_j is assigned to I_i in the allocation $\bar{x}$.

However, $\bar{p} \in F$ implies that $\bar{p}_k = 0$ for some k , and in this case $\bar{p}_k \geq -\alpha_{ik}$ for all i . If house H_j is assigned to I_i , then $u_i(H_j, \bar{p}_j) \geq u_i(H_k, \bar{p}_k)$ which is equivalent to $\alpha_{ij} + \bar{p}_j \geq \alpha_{ik} + \bar{p}_k \geq 0$ and obviously $\bar{p}_j \geq -\alpha_{ij}$. The theorem is now completely proved because $-\bar{p}_j = \bar{\pi}_j \geq 0$. ■

Chapter V. A FAIR WAGE-STRUCTURE

25. Introduction

In chapter III we have defined the meaning of a fair social state and thereby also the meaning of distributive justice. If we consider distributive justice as the goal of economic activity then the problem is to construct an economic system which generates fair social states. In this chapter we shall study one of the aspects of an economic system which is of great importance for the final distribution of welfare and that is the wage-system. We shall examine which claims can reasonably be imposed on a wage-system in an economy generating fair social states.

As a starting point for a definition of a fair wage-structure we can use the following. A wage-structure is fair if it is a part of an economic system generating fair social states. Such a definition is, however, not unique. Jobs and wages alone need not determine an individual's welfare level completely. There may exist other sources such as allowances for rent, subsidies for children, sickness benefits, pensions etc., which must also be considered. This means that any wage-structure could be part of a fair economy provided that that portion of individual welfare which can not be derived from wages be designed such that fair social states are generated. We must consequently interpret "part of a fair economy" in another way.

The wage-system is one of the distribution mechanisms in an economy and wages may play several roles. Besides being a structure for compensation for work done it can also be a part of the social security system and a part of the allocation system of an economy. A fair social state (X, g) is characterized by two properties that (X, g) be envy-free and that (X, g) be a Pareto-optimum with respect to ethical preferences. These two properties of a fair social state are supplied to different extents in the different distribution mechanisms of an economic system. Concerning the wage-system and the definition of

a fair wage-structure we shall consider that part of wages which is purely a compensation for work done. Fair wages will then be defined as those which secure the property of freedom from envy in a fair social state. We interpret this as a fair division of a cake consisting of a given sum of money, which has to be distributed as wages, and a given set of jobs.

This cake has to be equally divided in some sense and the resulting distribution of wages will be defined as a fair wage-structure.

To make clear in what sense an equal division of such a cake will support the envy-free property of a fair social state we consider the following simple example. A sum s of money has to be distributed fairly among n individuals according to our definition of a fair social state D1:10. We assume that at one moment t_0 all individuals are equally qualified for a part of the sum s of money available. We also assume that differences between the individuals may occur at another and later moment t_1 which could motivate an unequal distribution of s . To compensate for such an occurrence there is an insurance system. Each individual can use an arbitrary part of his share of s received at t_0 to insure himself and depending on the amount of that part, he will receive a certain sum of money if something were to occur at t_1 . The distribution system now consists of two parts. Firstly we divide the sum s among the n individuals and secondly we let each individual choose an insurance which fits the individual best. This distribution system will generate envy-free "social states" if the sum s is divided in equal shares at the moment t_0 . If suitable claims are imposed on the insurance system we also obtain fair social states at t_1 . Our wage-system supplying an equal division of wages and jobs, corresponds to the equal division of the sum s of money at the moment t_0 .

We have now given an account of what we shall mean by fair wages. An exact definition of this concept must be a suitable interpretation of these ideas. In the rest of the sections in this chapter we shall define in different

models fair wages which will also be called a fair wage-structure. In sec. 26, sec. 27, and sec. 28 we shall consider formal models of different kinds. These models are interpreted in sec. 29 and sec. 30 in non formal terms and the applicable definition of fair wages will be found in sec. 30. Finally we shall in sec. 31 compare our definition of fair wages from sec. 30 with some common wage-principles.

26. A Fair Wage-Structure in a Formal Model with Completely Indivisible Jobs

We shall now consider a very simple formal model of jobs and wages and define fair wages in that model. The formal model is, in principle, $\mathcal{G}_2$ from sec. 18 and $\mathcal{G}_3$ from sec. 21 where we interpret the indivisibles as jobs and the divisible good as the money which has to be distributed over the different jobs.

The models $\mathcal{G}_2$ and $\mathcal{G}_3$ will briefly be repeated.

- 1) There is a finite set $A = \{a_1, a_2, \dots, a_N\}$ of N indivisible jobs. A job $a \in A$ is completely defined when we have stated all those properties which characterize the particular job. One of these characteristics is the wage which is also the only one that can be changed. Therefore we describe a job with two coordinates $a = (e, p)$ where p is the wage.
- 2) There is a set of individuals $I = \{i_1, i_2, \dots, i_N\}$ where the number of individuals (N) is equal to the number of jobs. Each individual is assumed to have a preference relation, a complete, strictly monotone, and continuous pre-ordering, over the set of jobs $\{(e, p)\}$. Furthermore, he is assumed to demand only one element in A the choice of which depends on the wage p .
- 3) An allocation x is a function from the set of individuals I to the set of jobs: $x(i) = (e(i), p(i))$ for $i \in I$ i.e. we assign precisely one job to each individual. We also have a function $T : x \rightarrow p$ from the set of allocations to $\mathbb{R}^N$. By $T(x) = p$ we mean that the wage of job a_j is p_j in the allocation x . $T(x) = p$ is the wage-structure of the allocation x . We have also considered a set F of feasible allocations. F may be defined in different ways. If a given amount α of money is available then feasible allocations x

are those where $\sum_{j=1}^N p_j \leq \alpha$ for $T(x) = p$. Other types of restrictions can also be put on the wage-structure e.g. $p \geq 0$ or $f(p) \leq \alpha$ for some strictly increasing and continuous function $f : \mathbb{R}^N \rightarrow \mathbb{R}$ (compare A4:21).

We now have to define what will be meant by a fair wage-structure. We interpret the feasibility set F as the set of possible divisions of a cake of wages and jobs and possible assignments of the pieces to the individuals. The cake has to be divided equally according to some criterion and the resulting distribution of wages will be defined as a fair wage-structure. The criterion we use is the fairness criterion and according to that an allocation x is fair if

- i) x is Pareto-optimal in the feasibility set and
- ii) there is no envy in x i.e. no individual $i \in I$ would strictly prefer a job $x(j)$ to his own job $x(i)$ if $i \neq j$.

The first part i) of the fairness criterion secures that as large a cake as possible has been divided and the second part ii) secures that everybody has been treated equally in the division.

In a fair wage-structure all jobs have not necessarily been assigned the same wage (p_j equal). $p_1, p_2, \dots, p_N$ is a distribution of money such that the non-wage differences among the various jobs, are compensated in such a way that an individual considers his own job to be at least as good as any other job.

A fair wage-structure p can also be considered as an equilibrium price vector where the supply and demand of jobs is balanced. It also follows from the definition of a fair wage-structure that there is no objective way of measuring the "value" p_j of the job a_j . This kind of value, rather, depends on the individual preferences concerning the jobs.

Two questions are closely related to the definition of a fair wage-structure. Does a fair wage structure exist? Is a fair wage structure necessarily

unique? It is easy to find cases where several different fair wage-structures exist. Therefore, we have to consider a fair wage-structure only as a necessary condition for fairness. In cases where individuals' preferences are very different the set of fair wage-structures will in general be great. In the case where all individuals have exactly the same preferences only one fair wage-structure exists.

Concerning the existence of fair wage-structures we can use T1:18 and T4:21 in which the existence of fair allocations is proved under different conditions.

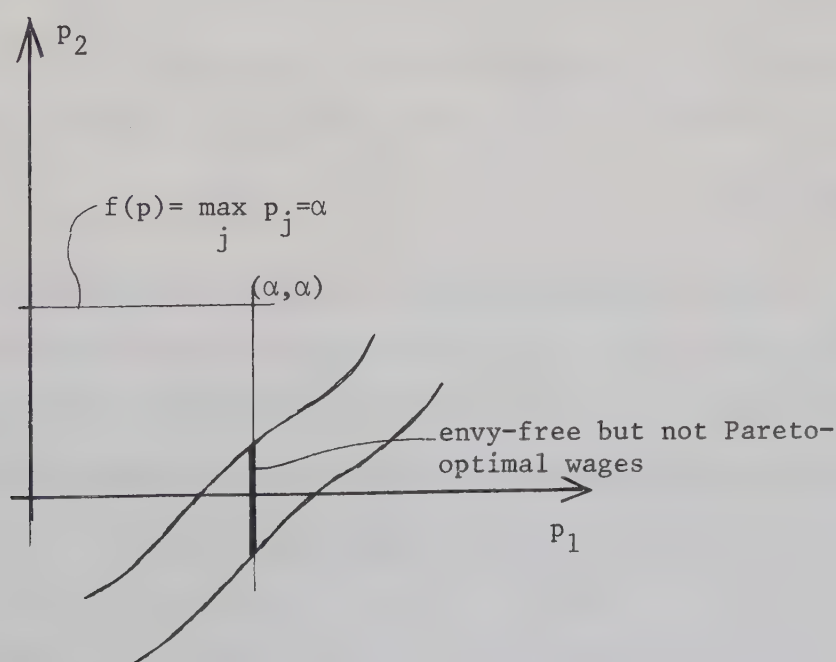
In $\mathfrak{Z}_2$ (T1:18) we have considered a case where feasible wage-structures $p \in R^N$ are those where $\sum p_j = \alpha$ i.e. the wage-structure belongs to an hyperplane in R^N . This means that there is a certain amount α of money which has to be distributed as wages among the jobs. In T1:18 a major presumption for the existence of fair allocations is the existence of a wage-structure p^i for $i \in I$ such that individual i considers all jobs (e_j, p_j^i) $j = 1, 2, \dots, N$ as equally desirable. He is indifferent to them. This condition is satisfied (T5:19) if e.g. any job (e_k, p_k) can be the preferred one in the set $\{(e_j, p_j) : j = 1, 2, \dots, N \text{ and } \sum p_j = x\}$ for a suitable choice of the various wages p_j ; p_k large and p_j , $j \neq k$ small. In many cases this is not a very serious restriction. If we, however, also want the wage-structure to include only positive elements ($p_j > 0$) we have to make additional assumptions. According to T1:21 it is sufficient for the existence of a positive wage-structure to assume the indifference-wages p^i to be positive i.e. $p_j^i > 0$ for all j and for all $i \in I$. As before such wage-structures (p^i) exist if any job (e_k, p_k) can be the preferred one in the set $\{(e_j, p_j) : j = 1, 2, \dots, N, \sum p_j = \alpha, p_j > 0\}$ for a suitable choice of the different wages p_j .

Cases where $p_j \rightarrow 0$ imply a very low utility level for the individual holding (e_j, p_j) , e.g. starvation, an assumption of the existence of such p^i is natural. In other cases, where e.g. the source of income is not only wages,

an assumption $p^i > 0$ may be violated and a fair wage-structure may possess negative elements.

The existence of a fair wage-structure is also proved in cases where feasible wage-structures lie in a more general set than a hyperplane. In T4:21 ($\mathcal{E}_3$) we allow the feasibility set to be a monotone and continuous deformation of a hyperplane. Feasible wage-structures $p \in \mathbb{R}^N$ are those where $f(p) \leq \alpha$ for some real continuous function f which also is strictly monotone in each coordinate. Let us consider two extreme examples of this case.

First, we have a case of maximum wages i.e. feasible wage-structures $p \in \mathbb{R}^N$ are those where $f(p) = \max_j p_j \leq \alpha$. No wages are allowed to exceed the number α . T4:21 cannot be used to prove the existence of fair wages because $f(p)$ is in this case not strictly increasing in each coordinate but only increasing. From the corollary of T4:21 follows in any case the existence of envy-free allocations and we discuss only this property in the following. It is also easy to see that fair allocations may not exist in cases where $f(p) = \max_j p_j$ or $f(p) = \min_j p_j$ defines the feasibility set. A figure will illustrate this.



For the existence of an envy-free wage-structure in this case (T4:21), the existence of envy-free wage-structures p' and p'' such that $f(p') \leq \alpha \leq f(p'')$ is presumed. To find such p' and p'' we can use the original existence theorem T1:21. It is also easy to prove that sufficient conditions for the existence of p' and p'' are the existence of indifferences p^i , $i \in I$ $((e_j, p_j^i) \sim_i (e_k, p_k^i) \forall j, k)$ with $f(p^i) \leq \alpha$.

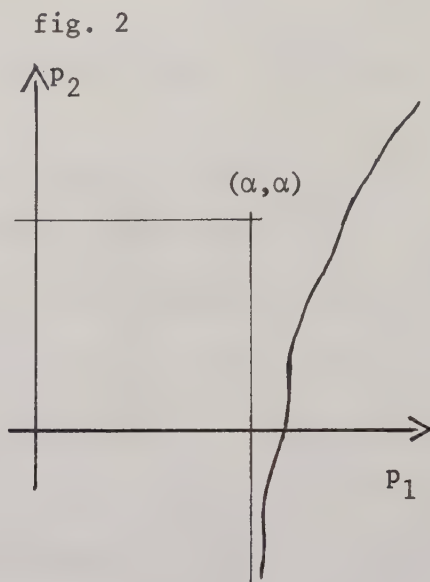
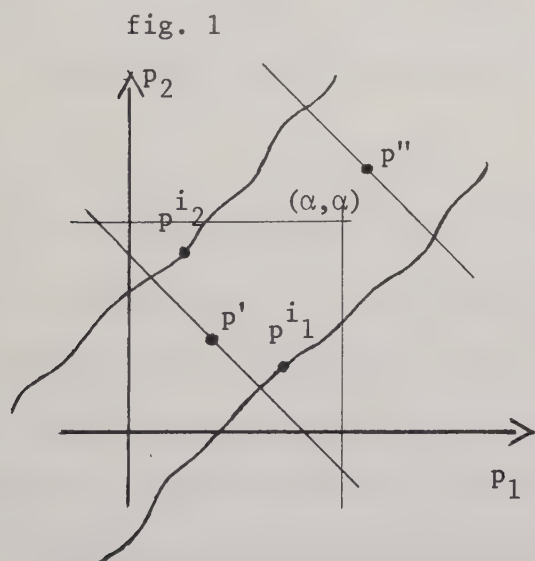


Fig. 1 illustrates the situation when p' and p'' always exist while fig. 2 shows a case where no indifferences p^i can be found with $f(p^i) = \max_j p_j^i \leq \alpha$. This nonexistence case can be interpreted as a case where job a_1 is so undesirable that a_2 is chosen at any price (p_2) if $p_1 \leq \alpha$. That situation seems not very realistic.

Now consider the case of minimum wages. We have as feasible wage-structures those $p \in \mathbb{R}^N$ for which $f(p) = \min_j p_j \geq \alpha$. Envy-free wage-structures will exist under the same conditions as in the previous case ($f(p) = \max_j p_j$). Therefore we omit the first part of the analysis of existence and consider instead the figures which correspond to fig. 1 and fig. 2.

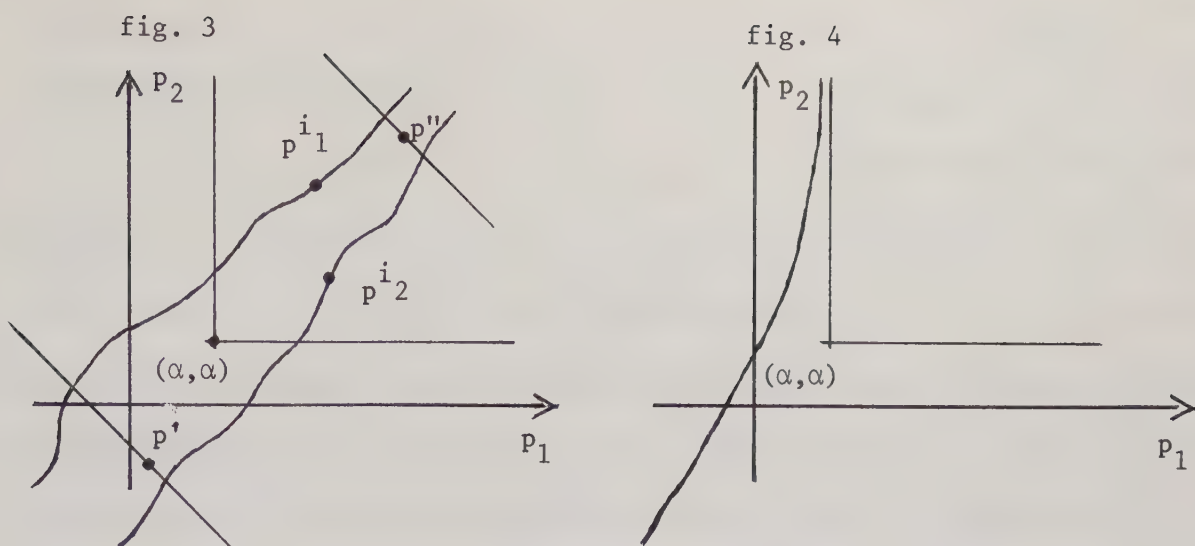


Fig. 3 illustrates the situation when p' and p'' always exist implying also the existence of an envy-free wage-structure. An individual with preferences according to fig. 4 will always consider the job (e_1, p_1) as strictly better than (e_2, p_2) if $p_j \geq \alpha$, $j = 1, 2$. This means that if there are two individuals with this sort of preferences no envy-free wage-structure can exist. This also fulfils the requirement $p_j \geq \alpha$, $j = 1, 2$. In the consideration of maximum wages the corresponding situation was not a very realistic one but there seems to be nothing strange about preferences in this relationship. If the job a_2 is very bad it will, regardless of how much the job is paid, never be as desirable as a_1 provided that this job is paid at least α .

Thus, a rule about minimum wages may very well be in conflict with a requirement of fair wages if the minimum level is too high.

27. Fair Wages when Preferences are Incomplete

In the models $\mathcal{E}_2$ and $\mathcal{E}_3$ in chapter IV preferences were assumed to be complete. A case with incomplete preferences was considered in sec. 22 ($\mathcal{E}_4^*$). The model $\mathcal{E}_4$ is the same as $\mathcal{E}_2$ except for the incompleteness of preferences. Clearly, however, in order to be able to make a reasonable definition of fairness, preferences cannot be too incomplete. Therefore we have assumed in $\mathcal{E}_4^*$ that

each individual $i \in I$ can be associated with a set $m_i \subset E$ (see A1:22) of jobs where m_i has certain properties.

A set m_i is interpreted as those jobs which i is capable of evaluating and for which he is qualified.

Preferences are assumed to be complete in the set $m_i \times R$ i.e. all jobs in m_i can be compared for all possible wages by individual i . If preferences are represented by a utility function then the meaning of this assumption is that if the utility level of one job $(e_j, \bar{p}_j)$ with wage $\bar{p}_j$ can be determined then the utility level of (e_j, p_j) can be determined for all $p_j \in R$.

Because of the incompleteness of preferences the definition of a fair wage-structure becomes weaker than before. The starting-point is the same; we want to divide a cake of money and jobs fairly among the individuals using the fairness criterion with the modification that one individual $i \in I$ can only envy another individual $j \in I$ if j has a job (e_k, p_k) with $e_k \in m_i$. Thus elements which do not belong to the individual's own qualification set m_i cannot evoke any envy in i .

The formal definition is:

An allocation of jobs $x: I \rightarrow E \times R$ is fair if 1) x is Pareto-optimal and 2) for all i : $x(i) \succsim_i x(j)$ for all $x(j) = (e_k, p_k)$ with $e_k \in m_i$.

A fair wage-structure is defined to be the distribution $p = (p_1, \dots, p_N)$ of money corresponding to a fair allocation.

It is now obvious that a fair wage-structure is a much weaker concept in the case of incomplete than in the case of complete, preferences. If e.g. $N = 2$ and $m_1 = \{e_1\}$ $m_2 = \{e_2\}$ then any wage-structure $p = (p_1, p_2)$ is fair because i_1 is only qualified for $a_1 = (e_1, p_1)$ and i_2 only qualified for $a_2 = (e_2, p_2)$ and no envy can exist. Thus in the case of incomplete preferences there may be much more space for further restrictions on the wage-structure. A fair

wage-structure is only a necessary condition for real fairness.

The existence of a fair wage-structure in $\mathcal{E}_4$ is proved under different assumptions. First we have to assume (A2:22) the existence of a distribution of jobs among the individuals so that each gets a job for which he is qualified i.e. belongs to his set m_i . The existence of a bijection $\varphi : I \rightarrow E$ such that $\varphi(i) \in m_i$ for all $i \in I$ is necessary.

The proof of existence of a fair wage-structure requires additional assumptions to compensate for the incompleteness of preferences. If $f^i : R \rightarrow \bar{R}^N$ is the generalized indifference function (see sec. 19), i.e. the curve in $\bar{R}^N$ which defines wage-structures corresponding to indifference between all jobs $((e_j, p_j) \sim_i (e_k, p_k) \forall j, k)$, of individual $i \in I$ then we have assumed (A4, A5 or A6) that all $f^i(t)$ pass through a common point in $\bar{R}^N$ ($\bar{R} = [-\infty, +\infty]$). With this additional assumption the existence of fair wage-structures can be proved (T1:22, T2:22).

We will mention the economic significance of this assumption only in one case viz. that the common point is $(0, \dots, 0)$. This case was also considered in the previous section where a positive wage-structure was illustrated. The motivation of the assumption from that section is of course still relevant. In $\mathcal{E}_4$ the assumption $f^i(t) = 0$ for some t secures the existence of a fair wage-structure which furthermore is positive.

28. A Fair Wage-Structure in a Formal Model where Each Job is Associated with a Set of Variable Characteristics

In the two cases from sec. 26 and sec. 27 we have considered jobs as totally indivisible in the meaning that an individual can take only one job. A job may, however, have divisible properties. For example, in sec. 26 and sec. 27, wage was such a property. In many cases a job has several variable properties such as working-hours, degree of dirtiness, degree of responsibility etc. Moreover, a job can also be characterized by properties of the particular

worker holding it. Thus, the more skillful or qualified worker may produce more than the less skillful or some workers may want to work harder than others and so on. In such cases a model with only one variable characteristic, the wage, is too crude. It may be desirable not only to value the job as an entirety but also to value the different characteristics in a fair way. What is a fair overtime compensation or what are fair differences in wages for the same jobs held by differently trained individuals? It is not possible to answer questions of these kinds from the models ξ_2 , ξ_3 or ξ_4 but in the model ξ_5 , defined in sec. 23, we take these aspects into consideration.

The main features of ξ_5 are the following:

- 1) Each job $a \in A$ can be associated with a set $m(a, q) \subset C$ of feasible characteristics of a . C is the set of all characteristics needed to describe a job. C is assumed to be a subset of R^n . The coordinate q is a real number and is supposed to be a measure of the "resources" (money) assigned to a . An increase in q may be used to raise the wage by the same amount or to decrease the working-hours or to improve the conditions of the job etc.
- 2) $I = \{i_1, i_2, \dots, i_N\}$ is the set of N individuals. Each individual has preferences, a complete strictly monotone and continuous pre-ordering over the set C of characteristics.
- 3) An allocation x is a function from the set I of individuals to the set C of characteristics such that all $x(i)$ belongs to different sets $m(a_j, q_j)$ i.e. two individuals have not been assigned the same job.

With each allocation x can also be associated the distribution of the various characteristics. This distribution can be defined by a function T from the set of allocations to $R^N \times R^n$. If $T(x) = p$ then p_{jk} is equal to the measure of characteristic k of job j . p_{j1} $j = 1, 2, \dots, N$ may be the distribution of wages i.e. the wage-structure.

By the set F of feasible allocations we mean $F =$

$= \{x : x(i) \in m(\varphi(i), q(\varphi(i))) \quad \forall i \in I \text{ for some bijection (assignment func-}$

tion) $\varphi : I \rightarrow A$ and $\sum_i q(\varphi(i)) \leq \alpha$. We interpret this to mean that there is a given amount α of the resource to distribute among the different jobs.

We are interested primarily in what will be meant by fair wages but in this model there is no logical reason to distinguish between one of the characteristics, such as that one which defines wages, and others. This means that fair wages have to be defined as a part of an allocation where all characteristics are distributed fairly. We consider the set F as the set of possible divisions of a cake which includes all those properties which define a job. A fair division of this cake will define not only a fair wage-structure but also a fair distribution of the other variable characteristics of a job. By a fair division we mean an allocation x which is fair according to the fairness criterion i.e. x is Pareto-optimal and no individual i with job $x(i)$ envies any other individual j with job $x(j)$.

It is shown in sec. 23 that in $\mathcal{E}_5$ it is possible to define a stronger concept of justice than the one defined by the fairness criterion. We have defined a fair equilibrium allocation x as an allocation where each individual i is associated with a subset S_i of C and where $x(i) \in S_i$ and is a most preferred element in $\bigcup_{i \in I} S_i$. It is obvious that it is the envy-free property of a fair allocation that has been strengthened. However, if furthermore $S_i = \{m(\varphi(i), q(\varphi(i)))\}$ for some resource distribution q with $\sum_{a \in A} q(a) = \alpha$ and for some assignment function φ then a fair equilibrium allocation in $\mathcal{E}_5$ also is Pareto-optimal and thereby fair (T1:23).

The advantage in considering fair equilibrium allocations in $\mathcal{E}_5$ instead of merely fair allocations is that in the former a theory of "marginal justice" is also obtained. These matters are discussed at the end of sec. 23. The significance of "marginal justice" is that to each job $x(i)$ in a fair equilibrium allocation x can be associated a vector $p_{x(i)}$ of relative prices of the different characteristics of the job $x(i)$. The different coordinates of $p_{x(i)}$ will represent over-time compensation, the differences of wages of

more or less qualified workers etc. When marginal changes of the characteristics of $x(i)$ are made and these changes are compensated according to $p_{x(i)}$ then the fairness of the allocation remains. Thus, all are of one mind about the correctness of the marginal prices $p_{x(i)}$ of the different job characteristics. We must note, however, that the same characteristics may have different prices depending on what job is changed. If we had had ordinary commodities instead of characteristics the marginal price would have been the same for one commodity regardless of the commodity bundle $x(i)$ to which it belonged.

Sufficient conditions for the existence of a fair distribution of characteristics and thereby also a fair wage-structure are also analysed in sec. 23. The exact conditions are those (A1, A2, A3 and A5 in sec. 23) which define ξ_5 .

29. Fair Wages in a Non Formal Static Model

In the previous sections of this chapter we have interpreted the formal models from Chapter IV in jobs-wages terms. The models are, however, still formal and we have only put new names on the variables and discussed the relevance of the assumptions made in the light of an interpretation of the symbols as jobs and wages.

We shall now discuss these concepts in non formal terms and thereby, hopefully, reach a more applicable definition of fair wages in real situations than the previous ones. The formal models will, of course, serve as a basis for our discussion and will also indicate under which conditions fair wages may exist.

In this section we shall consider a static situation on the labour market. A generalization to a dynamic situation (redistribution of jobs are permitted) will be made in the next section.

The wage-system has to fulfil several aims within an economic system. It may

serve as the allocation instrument, or a part of it, for jobs; it may be a part of the social security system and so on. In our discussion of fair wages we have only considered that part of wages which can be considered as the pure compensation for work done i.e. wages compensate for differences in jobs. This will also be the interpretation of wages in the following discussion.

The utility of a job (a, p) can be separated into two parts. These are the utility or disutility of performing the job itself and the potential utility connected with the set of choices or budgetset which is at least partly defined by the wage p . The money p received for the job a results in utility indirectly by increasing the individual's possible choices of commodities, services and so on. This means that p must be a relevant part of the measure of this set. If e.g. taxes are imposed on incomes then the relevant measure of wages are wages before taxation because some commodities may be paid for from the income before taxation. In Sweden an example of this is the rent costs of a privately owned single-family dwelling.

The wage p will also be considered as the momentary reward for the job a e.g. earnings per week, per month, or per year.

By a job a we shall mean primarily a profession; carpentry, teaching, medicine, etc. i.e. the profession for which one has been trained. Jobs, interpreted in this way, correspond to jobs in the formal models $\mathcal{G}_2$, $\mathcal{G}_3$ or $\mathcal{G}_4$ (sec. 26, 27) i.e. we are not considering any variable characteristics of a job except for wages. In that case wages have to be interpreted as average-wages for the particular profession.

For a more detailed picture of a job we can use the concepts from $\mathcal{G}_5$ where a job is characterized by certain properties. Thus we can also distinguish between different jobs within the same profession e.g. teachers of mathematics of different ages, with different working-hours, placed at different

schools etc. These differences within the same profession, teaching, may justify marginal differences in wages. Thus we may expect two theories of fair wages which could be characterized as a macro and a micro theory. The dividing-line between these two concepts are, of course, not exactly defined but it can be practical to use such a division.

Now, consider a static situation (or a one period case). There is a given number of individuals who are going to make their occupational choices. We assume that each individual has to go through some kind of education or training before he can begin working and that the number of possible education or training situations is the same as the number of individuals.

We also assume that the occupational choice is final i.e. the individual is allowed to choose a profession only once and when he gets a job he has to keep it for the rest of the period.

Wages are also assumed to be constant for the entire period which is to be considered.

Individuals have preferences over different professions and the actual occupational choice depends on the wages.

In accordance with the definitions of fair wages from the previous sections we now define a fair wage-structure, or briefly fair wages, as those wages which result in equilibrium on the educational market.

We interpret this as a fair division, according to the fairness criterion, of a cake consisting of a certain amount of money and different professions. Fair wages have the effect of compensating for differences in the different professions.

This is the macro model interpretation ($\mathcal{E}_2$, $\mathcal{E}_3$, or $\mathcal{E}_4$). If we, furthermore, add the micro model ($\mathcal{E}_5$) then we have a definition of fair wages for different jobs within the same profession and also a definition of fair differences

in wages between different qualified individuals having "the same" job. In that case fair wages are those which are part of a fair allocation (or a fair equilibrium allocation) of the characteristics of a job according to the fairness criterion.

The conditions necessary for the existence of fair wages are discussed in the previous sections (sec. 26, 27, 28) and we will not repeat this discussion in general. Some of the assumptions, however, require further comment.

Firstly, we have assumed that jobs can be described as indivisibles. This is interpreted to mean that an individual prefers holding one single job during the period considered rather than holding several jobs. When a profession requires long training such an assumption may be natural because of the great costs of changing profession. In other cases we can justify the assumption by assuming the period considered to be short; it is not so simple to change jobs at short intervals.

Secondly, we have assumed the individuals to have preferences with certain properties over the set of jobs. A delicate point is whether preferences can be assumed to be complete. The discussion in sec. 27 showed that the definition of fairness becomes weaker the more incomplete preferences are. When defining fair wages we have considered the situation when an individual makes his occupational choice. Essential for preferences to be complete in this situation seems to be that two conditions be satisfied. The individual must have complete information about existing professions and he must be qualified or at least be able to train himself for any profession. Concerning the condition of complete information about professional opportunities the individual must at least know which professions exist. How much more information about a profession or a job he must know is a problem of optimality. We may consider information about a profession as one of the properties which characterizes the profession and that there is a cost in producing this information. Thus, there must be an optimal distribution of resources among the

different characteristics of a profession. The claim of optimality is a part of the definition of fairness!

Complete preferences also required the individual to be "qualified" for any job or education. This condition can be satisfied only under very special conditions. All individuals have to be talented and have no special reductions in their physical and mental capacities. The best approximation of such a situation is, perhaps, the situation when young people for the first time are going to choose a job, a profession or an education. Most individuals have then, at least theoretically, a very large set of possible choices. However, some professions always require special talents; anybody can probably not be a world champion at tennis or a great scientist etc. The conclusion that can be drawn is that depending on an individual's "personal goods" a condition about qualification can be more or less satisfied. The degree of qualification may not only vary from individual to individual but also depend on the age of a particular individual. An old person is, perhaps, not able to do hard physical work.

Thus, it seems to be necessary to assume preferences not to be complete. The incompleteness of preferences weakens the criterion of justice but in those cases where preferences are nearly complete the weakness is not serious.

30. Fair Wages in a Non Formal Dynamic Model

An unrealistic restriction in the model discussed in the previous section is the requirement that the occupational or job choice be final and no changes be allowed. This condition will now be relaxed and we shall consider a case where the time is divided into periods. We shall assume that the static model from the previous section is relevant in each period and changes of professions or jobs are allowed at the beginning of each period. We also allow an individual's preferences to change from one period to another and furthermore the set of individuals in one period must not be the same as the set of

individuals in other periods. This implies that at the end of a period some of the individuals may leave the labour market and in the beginning of the next period some other individuals may enter the market.

Now we mean by a dynamic fair wage-structure, or briefly fair wages, a set of wage-structures, one in each period, such that the wage-structure in a period is fair according to the definition in the static model in sec. 29.

Some consequences of the definition are easily derived. As we remarked earlier the strength of the definition depended on to what extent preferences were complete. In this dynamic situation we have e.g. individuals of different ages implying also that the degree of completeness of preferences may vary greatly among individuals. It seems natural to assume preferences of younger individuals to be more complete than those of older individuals. This means that the preferences of the younger ones will influence the fair wage-structure more than the preferences of the older ones. This may appear unjust, however everyone has once been young and therefore all have an equal opportunity to influence wages if we consider the wage-structure which an individual meets during his whole life. Accordingly, there is no internal conflict with the envy-free property of a fair social state.

Another consequence of the definition of fair wages is the great opportunity of choosing or changing a job, a profession, or an education at the beginning of each period. There may e.g. not be any queues or barriers to the training situations. The stipulation of "no queues" implies that the necessary waiting time for a job not exceed the length of any one time period considered.

Imbalance in some of the markets mentioned is always a sign of unfair wages in some sense. It is, however, not always possible to remove the unfairness simply by changing the wage-structure. In the definition of fair wages we have assumed, in the case of complete preferences, that there is an equal

number of individuals and of jobs. When preferences were incomplete a corresponding presumption for fair wages was the existence of an assignment of individuals to jobs for which they were qualified (or had preferences over). If these conditions are not satisfied then there are imbalances for any wage-structure. This means that one has to be careful when recommending changes of wages so as not to cause imbalance in some market. If some group of jobs for some reason (tradition) are pointed out for a special group of individuals (women) and if the number of jobs is less than the number of individuals in the group then there must necessarily be some imbalance. In such cases the number of jobs for the group must be increased in some way. One solution may be to consider "to be unemployed with compensation" as a job. We have still to be careful because the individuals of the group have incomplete preferences and this will imply that the average wage of the group can exist on many different levels in a fair wage-structure. The incompleteness of the preferences still weakens the criterion of justice.

31. A Comparison between Fair Wages and Some Common Principles of Setting Wages

In the wage-political debate in Sweden¹ several principles according to which wages might be set are discussed. One principle is that wages ought to be a function of some kind of "work valuation" or depend on the qualifications of the worker. On the other hand pure market wages is another possible principle by which wages may be set as is a principle whereby wages depend on the needs of the worker. Our definition of fair wages takes these principles into account to some extent.

By a "work valuation" we mean a complete description of a particular job. This is, of course, the same as our description of a job as a point in the

¹ See e.g. Meidner, Öhman (1972), Fackföreningsrörelsen...(1951), Lönepolitiken inom...(1963), or Lönepolitik. (1971).

set of characteristics. A "work valuation" is the empirical measurement of these characteristics (except for wages). A "work valuation" alone is not sufficient to determine wages and there must be some supplementary principle. If one job is more demanding in every characteristic than another job then this also implies that the first job be paid better than the second one. This is a simple principle which complements a "work valuation". When talking about "work valuation" we also include the principle of monotonicity. A "work valuation" in this extended form is, however, not sufficient to determine wages. Only an incomplete ranking of jobs is obtained. On the other hand a "work valuation" will now tell us that equal jobs have to be paid equally. Wages are e.g. not permitted to depend on the company in which the job is situated or to depend on sex, race, nationality etc.

Our definition of fair wages includes a principle of "work valuation". First we may interpret the empirical measurement of the different characteristics of a job as a means to fulfil the claim of optimal allocation of characteristics in a fair wage-structure. It is a condition of optimal information which has to be satisfied.

The monotonicity principle is also satisfied in a fair wage-structure since it is a consequence of the envy-free property of a fair allocation of jobs.

Wages may also depend on the qualifications of the workers. A fair wage-structure takes such a principle into account. The qualifications of a worker is one of the characteristics which defines a job and consequently we are only interested in those qualifications which actually may influence the job-product.

A theory for the valuation of different types of qualifications is provided by our model where jobs are associated with a set of variable characteristics (sec. 28). A fair wage-structure also includes fair marginal wages of the different characteristics and the degree of qualification is one of these

characteristics. The marginal wages are fair in e.g. the following sense. If we first have two equally qualified workers who have the same job then fair wages require that wages be equal. If later one of the workers by education becomes more qualified for the job and therefore also produces more then there is an additional compensation to the more qualified worker which does not disturb the envy-free property of the fair wage-structure.

In many cases the wages of the same jobs may differ depending on the time that the worker has held the job; the longer the period of time, the higher the wages. Such a wage policy may be justified if the skill of the worker increases with the time he has held the job. A profession which claims long education is an extreme case of this.

During the education a person does not produce anything and since the process of education is costly the net production is negative. This means that in some way or another the person has to pay back these costs. One possibility is that he pay for his education himself and another is that he does not receive full payment for the job until he has held the job for some time. In this way there is a cost for the person to change professions too quickly if the profession requires a long education.

Fair wages may also be compared with equilibrium wages on the labour market or briefly market wages. We shall point out some differences between these two concepts.

Now consider a job which necessitates a lengthy education. If at some time the demand for qualified workers exceeds the supply then the momentary balance can be achieved only by raising wages. In that case the supply of workers is constant but the demand will decrease. An effect of the increased wages will probably also be an increase in the demand for this kind of educational opportunities and if the number of these is fixed then we may have an imbalance on that educational market.

Fair wages, on the other hand, requires primarily a balance in the educational market and a deficit in the supply of labour is not always sufficient reason to increase wages. Of course, in a situation of general equilibrium both the labour market and the educational market are balanced simultaneously but in a dynamic economy, where inertias of different kinds are at play, it may be difficult to achieve general equilibrium. In such a case one has to choose which kind of equilibrium will influence the wage-structure.

An essential difference between fair wages and market wages is that the former principle takes the preferences of more individuals into account than the latter when setting wages. Market wages take only the preferences of those individuals who for the moment are qualified for the jobs into account while fair wages depend on the preferences of all. This point is, of course, quite decisive for the definition of fairness. We have earlier remarked that the fairness criterion is weakened when preferences are incomplete and for that reason market wages are not satisfactory as a definition of fair wages.

Another difference between fair wages and market wages can be pointed out. Suppose there is a group of individuals who can take only a special category of jobs and no education is needed for these jobs. If for a certain level of wages the demand for jobs exceeds the supply then a market wage policy would recommend a reduction of wages. In a situation where the supply of jobs is not influenced by a reduction of wages balance is restored only if the number of individuals demanding this kind of jobs is decreased. The definition of fair wages, on the other hand, presupposes that the number of jobs and individuals are the same and wages are not primarily the instrument to create jobs.

Finally, let us consider the principle where wages depend on the needs of the workers. Our definition of fair wages does not explicitly take such a principle into account. The wage in our meaning is a number (money) which is assigned to a job and this assignment does not depend on who holds the job

and consequently also not on the needs of the worker. A fair wage-structure is a set of wages such that all jobs in some meaning are valued equally. On the other hand fair wages are supposed to be a part of an economy generating fair social states and the needs of an individual are of course taken into account in a fair state. This means that wages in a fair economy may depend on individuals' needs. This is also the case if it is the practical solution to the distribution problem in the economy. In that case fair wages are only a part of the actual wages and a part which is independent on who is holding a particular job.

BIBLIOGRAPHY

- Arrow, K.J. "Values and Collective Decision Making", in Laslett and Runciman, 1967.
- Arrow, K.J. "Some Ordinalist-Utilitarian Notes on Rawls's Theory of Justice." *The Journal of Philosophy* 70, 1973.
- Arrow, K.J. "Rawls's Principle of Just Savings." *Swedish Journal of Economics*, 1973.
- Arrow, K.J. and F.H. Hahn. *General Competitive Analysis*. Oliver & Boyd, Edinburgh, 1971.
- Buchanan, J.M. "A Hobbesian Interpretation of the Rawlsian Difference Principle." *Klyklos*, vol. 29, 1976.
- Daniel, T.E. "A Revised Concept of Distributional Equity." *Journal of Economic Theory* 11, 1975.
- Debreu, G. "Continuity Properties of Paretian Utility." *International Economic Review* 5, 1964.
- Dieudonné, J. *Foundations of Modern Analysis*. Academic Press, New York and London 1960.
- Dubins, L.E. and Spanier, E.H. "How to Cut a Cake Fairly." *American Mathematical Monthly* 68, 1961.
- Fackföreningsrörelsen och den fulla sysselsättningen. Rapport till LO-kongressen 1951, Stockholm.
- Fleming, M. "A Cardinal Concept of Welfare." *Quarterly Journal of Economics* 66, 1952.
- Foley, D.K. "Resource Allocation in the Public Sector." *Yale Economic Essays* 7, 1967.
- Gale, D. *The Theory of Linear Economic Models*. McGraw-Hill, New York 1960.
- Gärdenfors, P. *Fairness without Interpersonal Comparisons*. Working-Paper no. 15, August 1976. The Mattias Fremling Society, Kungshuset, Lundagård, Lund, Sweden.

- Halmos, P.R. Finite-dimensional Vector Spaces. Van Nostrand, Princeton, N.J., 1958.
- Harsanyi, J.C. "Cardinal Welfare, Individualistic Ethics and Interpersonal Comparisons of Utility." Journal of Political Economy 63, 1955.
- Hörmander, L. Integrationsteori. Föreläsningar ht 1959. Stockholms Universitets Matematiska Institution, 1964.
- Jaskold-Gabszewicz, J. "Coalitional Fairness of Allocations in Pure Exchange Economies." Econometrica, Vol. 43, 1975.
- Lancaster, K.J. "A New Approach to Consumer Theory." Journal of Political Economy, 1966.
- Luce, R.D. and H. Raiffa. Games and Decisions. Wiley, New York, 1957.
- Lönepolitik. Rapport till LO-kongressen 1971.
- Lönepolitiken inom tjänstemannarörelsen. TCO:s lönepolitiska rapport. Stockholm 1963.
- Meidner, R. och B. Öhman. Solidarisk lönepolitik. Tidens förlag, Stockholm 1972.
- Pattanaik, P.K. "Risk, Impersonality and the Social Welfare Function." Journal of Political Economy, 1968, December.
- Pazner, E.A. Pitfalls in the Theory of Fairness. Discussion Paper No. 181. October 1975. Northwestern University, Evanston, Illinois.
- Pazner, E. and D. Schmeidler. "A Difficulty in the Concept of Fairness." Review of Economic Studies 41, 1974.
- Phelps, E.S. Economic Justice, Selected Readings. Penguin Education, 1973.
- Rawls, J. A Theory of Justice. Oxford 1972.
- Schmeidler, D. and K. Vind. "Fair net trades." Econometrica 40, 1972.
- Sen, A.K. Collective Choice and Social Welfare. Oliver & Boyd, Edinburgh, 1971.
- Sen, A.K. On Economic Inequality. Clarendon Press, Oxford, 1973.
- Simmons, G.F. Introduction to Topology and Modern Analysis. McGraw-Hill Book Company. New York, 1963.

Suppes, P. "Some Formal Models of Grading Principles." *Synthese*, 6, 1966.

Varian, H.R. "Equity, Envy and Efficiency." *Journal of Economic Theory* 9, 1974.

Varian, H.R. "Distributive Justice, Welfare Economics, and the Theory of Fairness." *Philosophy and Public Affairs* 4, No 3, 1975.

Varian, H.R. Two Problems in the Theory of Fairness. Working Paper.

Department of Economics. Massachusetts Institute of Technology. 1975.

Lund Economic Studies

1. Guy Arvidsson: Bidrag till teorin för verkningarna av räntevariationer. Lund 1962.
2. Björn Thalberg: A Trade Cycle Analysis. Extensions of the Goodwin Model. Lund 1966.
3. Bengt Höglund: Modell och observationer. En studie av empirisk anknytning och aggregation för en linjär produktionsmodell. Lund 1966.
4. Alf Carling: Industrins struktur och konkurrensförhållande. Lund 1968.
5. Tony Hagström: Kreditmarknadens struktur och funktionssätt. Lund 1968.
6. Göran Skogh: Straffrätt och samhällsekonomi. Lund 1973.
7. Ulf Jakobsson och Göran Norman: Inkomstbeskattningen i den ekonomiska politiken. En kvantitativ analys av systemet för personlig inkomstbeskattning 1952–1971. Lund 1974.
8. Eskil Wadensjö: Immigration och samhällsekonomi. Immigrationns ekonomiska orsaker och effekter. Lund 1973.
9. Rögnvaldur Hannesson: Economics of Fisheries: Some Problems of Efficiency. Lund 1974.
10. Charles Stuart: Search and the organization of marketplaces. Lund 1975.
11. S. Enone Metuge: An input–output study of the structure and resource use in the cameroon economy. Lund 1976.
12. Bengt Jönsson: Cost–Benefit Analysis in Public Health and Medical Care. Lund 1976.
13. Agneta Kruse och Ann-Charlotte Ståhlberg: Effekter av ATP — en samhällsekonomisk studie. Lund 1977.
14. Krister Hjalte: Sjörestaureringens ekonomi. Lund 1977.
15. Lars-Gunnar Svensson: Social Justice and Fair Distributions. Lund 1977.